CHAPTER 1

Vector Analysis

Vector analysis is a mathematical subject that is better taught by mathematicians than by engineers. Most junior and senior engineering students have not had the time (or the inclination) to take a course in vector analysis, although it is likely that vector concepts and operations were introduced in the calculus sequence. These are covered in this chapter, and the time devoted to them now should depend on past exposure.

The viewpoint here is that of the engineer or physicist and not that of the mathematician. Proofs are indicated rather than rigorously expounded, and physical interpretation is stressed. It is easier for engineers to take a more rigorous course in the mathematics department after they have been presented with a few physical pictures and applications.

Vector analysis is a mathematical shorthand. It has some new symbols and some new rules, and it demands concentration and practice. The drill problems, first found at the end of Section 1.4, should be considered part of the text and should all be worked. They should not prove to be difficult if the material in the accompanying section of the text has been thoroughly understood. It takes a little longer to "read" the chapter this way, but the investment in time will produce a surprising interest. ■

1.1 SCALARS AND VECTORS

The term *scalar* refers to a quantity whose value may be represented by a single (positive or negative) real number. The x, y, and z we use in basic algebra are scalars, and the quantities they represent are scalars. If we speak of a body falling a distance L in a time t, or the temperature T at any point in a bowl of soup whose coordinates are x, y, and z, then L, t, T, x, y, and z are all scalars. Other scalar quantities are mass, density, pressure (but not force), volume, volume resistivity, and voltage.

A *vector* quantity has both a magnitude[1] and a direction in space. We are concerned with two- and three-dimensional spaces only, but vectors may be defined in

[1] We adopt the convention that magnitude infers absolute value; the magnitude of any quantity is, therefore, always positive.

n-dimensional space in more advanced applications. Force, velocity, acceleration, and a straight line from the positive to the negative terminal of a storage battery are examples of vectors. Each quantity is characterized by both a magnitude and a direction.

Our work will mainly concern scalar and vector *fields*. A field (scalar or vector) may be defined mathematically as some function that connects an arbitrary origin to a general point in space. We usually associate some physical effect with a field, such as the force on a compass needle in the earth's magnetic field, or the movement of smoke particles in the field defined by the vector velocity of air in some region of space. Note that the field concept invariably is related to a region. Some quantity is defined at every point in a region. Both *scalar fields* and *vector fields* exist. The temperature throughout the bowl of soup and the density at any point in the earth are examples of scalar fields. The gravitational and magnetic fields of the earth, the voltage gradient in a cable, and the temperature gradient in a soldering-iron tip are examples of vector fields. The value of a field varies in general with both position and time.

In this book, as in most others using vector notation, vectors will be indicated by boldface type, for example, **A**. Scalars are printed in italic type, for example, *A*. When writing longhand, it is customary to draw a line or an arrow over a vector quantity to show its vector character. (CAUTION: This is the first pitfall. Sloppy notation, such as the omission of the line or arrow symbol for a vector, is the major cause of errors in vector analysis.)

1.2 VECTOR ALGEBRA

With the definition of vectors and vector fields now established, we may proceed to define the rules of vector arithmetic, vector algebra, and (later) vector calculus. Some of the rules will be similar to those of scalar algebra, some will differ slightly, and some will be entirely new.

To begin, the addition of vectors follows the parallelogram law. Figure 1.1 shows the sum of two vectors, **A** and **B**. It is easily seen that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, or that vector addition obeys the commutative law. Vector addition also obeys the associative law,

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

Note that when a vector is drawn as an arrow of finite length, its location is defined to be at the tail end of the arrow.

Coplanar vectors are vectors lying in a common plane, such as those shown in Figure 1.1. Both lie in the plane of the paper and may be added by expressing each vector in terms of "horizontal" and "vertical" components and then adding the corresponding components.

Vectors in three dimensions may likewise be added by expressing the vectors in terms of three components and adding the corresponding components. Examples of this process of addition will be given after vector components are discussed in Section 1.4.

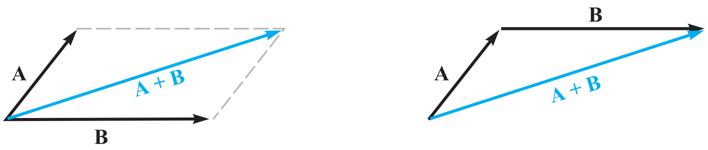

Figure 1.1 Two vectors may be added graphically either by drawing both vectors from a common origin and completing the parallelogram or by beginning the second vector from the head of the first and completing the triangle; either method is easily extended to three or more vectors.

The rule for the subtraction of vectors follows easily from that for addition, for we may always express $\mathbf{A}-\mathbf{B}$ as $\mathbf{A}+(-\mathbf{B})$; the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition.

Vectors may be multiplied by scalars. The magnitude of the vector changes, but its direction does not when the scalar is positive, although it reverses direction when multiplied by a negative scalar. Multiplication of a vector by a scalar also obeys the associative and distributive laws of algebra, leading to

$$(r+s)(\mathbf{A}+\mathbf{B}) = r(\mathbf{A}+\mathbf{B}) + s(\mathbf{A}+\mathbf{B}) = r\mathbf{A} + r\mathbf{B} + s\mathbf{A} + s\mathbf{B}$$

Division of a vector by a scalar is merely multiplication by the reciprocal of that scalar. The multiplication of a vector by a vector is discussed in Sections 1.6 and 1.7. Two vectors are said to be equal if their difference is zero, or $\mathbf{A} = \mathbf{B}$ if $\mathbf{A} - \mathbf{B} = 0$.

In our use of vector fields we shall always add and subtract vectors that are defined at the same point. For example, the *total* magnetic field about a small horseshoe magnet will be shown to be the sum of the fields produced by the earth and the permanent magnet; the total field at any point is the sum of the individual fields at that point.

If we are not considering a vector *field*, we may add or subtract vectors that are not defined at the same point. For example, the sum of the gravitational force acting on a 150 lb_f (pound-force) man at the North Pole and that acting on a 175 lb_f person at the South Pole may be obtained by shifting each force vector to the South Pole before addition. The result is a force of 25 lb_f directed toward the center of the earth at the South Pole; if we wanted to be difficult, we could just as well describe the force as 25 lb_f directed *away* from the center of the earth (or "upward") at the North Pole.[2]

1.3 THE RECTANGULAR COORDINATE SYSTEM

To describe a vector accurately, some specific lengths, directions, angles, projections, or components must be given. There are three simple methods of doing this, and about eight or ten other methods that are useful in very special cases. We are going

[2] Students have argued that the force might be described at the equator as being in a "northerly" direction. They are right, but enough is enough.

to use only the three simple methods, and the simplest of these is the *rectangular*, or *rectangular cartesian*, coordinate system.

In the rectangular coordinate system we set up three coordinate axes mutually at right angles to each other and call them the x, y, and z axes. It is customary to choose a *right-handed* coordinate system, in which a rotation (through the smaller angle) of the x axis into the y axis would cause a right-handed screw to progress in the direction of the z axis. If the right hand is used, then the thumb, forefinger, and middle finger may be identified, respectively, as the x, y, and z axes. Figure 1.2*a* shows a right-handed rectangular coordinate system.

A point is located by giving its x, y, and z coordinates. These are, respectively, the distances from the origin to the intersection of perpendicular lines dropped from the point to the x, y, and z axes. An alternative method of interpreting coordinate

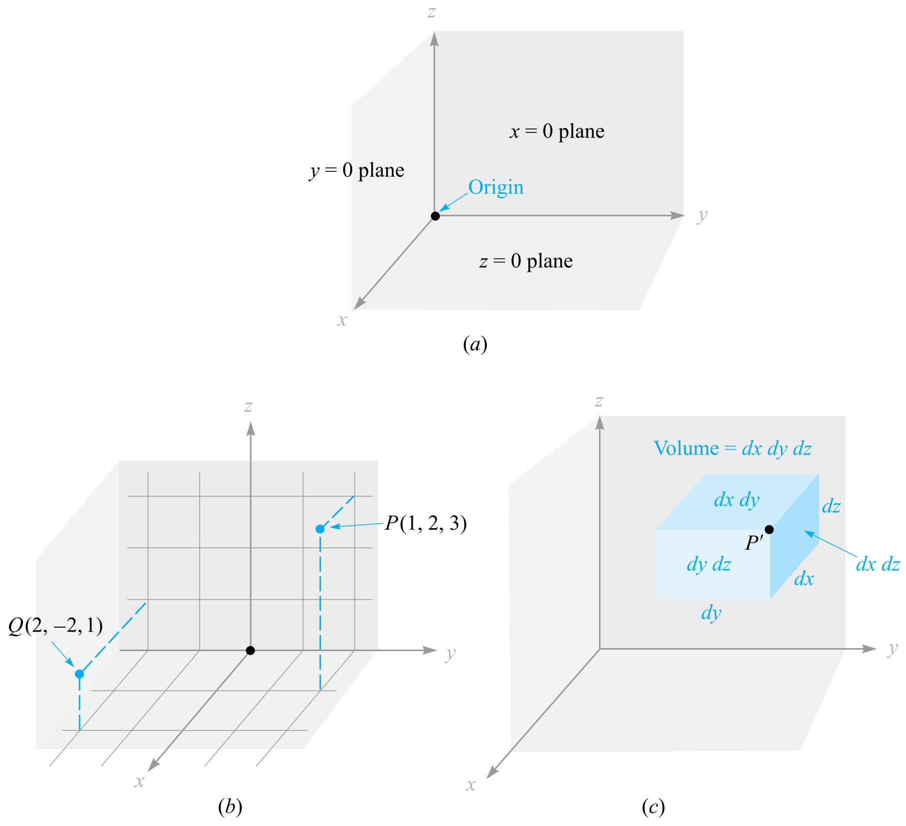

Figure 1.2 (*a*) A right-handed rectangular coordinate system. If the curved fingers of the right hand indicate the direction through which the x axis is turned into coincidence with the y axis, the thumb shows the direction of the z axis. (*b*) The location of points $P(1, 2, 3)$ and $Q(2, -2, 1)$. (*c*) The differential volume element in rectangular coordinates; dx, dy, and dz are, in general, independent differentials.

values, which *must* be used in all other coordinate systems, is to consider the point as being at the common intersection of three surfaces. These are the planes $x =$ constant, $y =$ constant, and $z =$ constant, where the constants are the coordinate values of the point.

Figure 1.2*b* shows points P and Q whose coordinates are $(1, 2, 3)$ and $(2, -2, 1)$, respectively. Point P is therefore located at the common point of intersection of the planes $x = 1$, $y = 2$, and $z = 3$, whereas point Q is located at the intersection of the planes $x = 2$, $y = -2$, and $z = 1$.

As we encounter other coordinate systems in Sections 1.8 and 1.9, we expect points to be located at the common intersection of three surfaces, not necessarily planes, but still mutually perpendicular at the point of intersection.

If we visualize three planes intersecting at the general point P, whose coordinates are x, y, and z, we may increase each coordinate value by a differential amount and obtain three slightly displaced planes intersecting at point P', whose coordinates are $x + dx$, $y + dy$, and $z + dz$. The six planes define a rectangular parallelepiped whose volume is $dv = dxdydz$; the surfaces have differential areas dS of $dxdy$, $dydz$, and $dzdx$. Finally, the distance dL from P to P' is the diagonal of the parallelepiped and has a length of $\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$. The volume element is shown in Figure 1.2*c*; point P' is indicated, but point P is located at the only invisible corner.

All this is familiar from trigonometry or solid geometry and as yet involves only scalar quantities. We will describe vectors in terms of a coordinate system in the next section.

1.4 VECTOR COMPONENTS AND UNIT VECTORS

To describe a vector in the rectangular coordinate system, let us first consider a vector $\mathbf{r}$ extending outward from the origin. A logical way to identify this vector is by giving the three *component vectors*, lying along the three coordinate axes, whose vector sum must be the given vector. If the component vectors of the vector $\mathbf{r}$ are $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, then $\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. The component vectors are shown in Figure 1.3*a*. Instead of one vector, we now have three, but this is a step forward because the three vectors are of a very simple nature; each is always directed along one of the coordinate axes.

The component vectors have magnitudes that depend on the given vector (such as $\mathbf{r}$), but they each have a known and constant direction. This suggests the use of *unit vectors* having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values. We reserve the symbol $\mathbf{a}$ for a unit vector and identify its direction by an appropriate subscript. Thus $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the unit vectors in the rectangular coordinate system.[3] They are directed along the x, y, and z axes, respectively, as shown in Figure 1.3*b*.

If the component vector $\mathbf{y}$ happens to be two units in magnitude and directed toward increasing values of y, we should then write $\mathbf{y} = 2\mathbf{a}_y$. A vector $\mathbf{r}_P$ pointing

[3] The symbols $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are also commonly used for the unit vectors in rectangular coordinates.

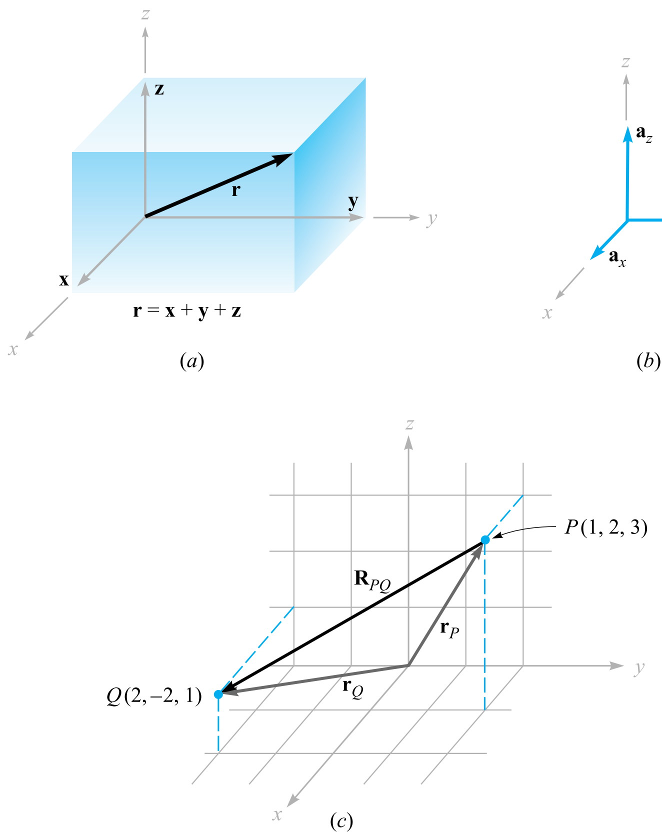

Figure 1.3 (*a*) The component vectors **x**, **y**, and **z** of vector **r**. (*b*) The unit vectors of the rectangular coordinate system have unit magnitude and are directed toward increasing values of their respective variables. (*c*) The vector $\mathbf{R}_{PQ}$ is equal to the vector difference $\mathbf{r}_Q - \mathbf{r}_P$.

from the origin to point $P(1, 2, 3)$ is written $\mathbf{r}_P = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$. The vector from P to Q may be obtained by applying the rule of vector addition. This rule shows that the vector from the origin to P plus the vector from P to Q is equal to the vector from the origin to Q. The desired vector from $P(1, 2, 3)$ to $Q(2, -2, 1)$ is therefore

$$\begin{aligned} \mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P &= (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z \\ &= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z \end{aligned}$$

The vectors $\mathbf{r}_P$, $\mathbf{r}_Q$, and $\mathbf{R}_{PQ}$ are shown in Figure 1.3*c*.

The last vector does not extend outward from the origin, as did the vector **r** we initially considered. However, we have already learned that vectors having the same magnitude and pointing in the same direction are equal, so we see that to help our visualization processes we are at liberty to slide any vector over to the origin before determining its component vectors. Parallelism must, of course, be maintained during the sliding process.

If we are discussing a force vector **F**, or indeed any vector other than a displacement-type vector such as **r**, the problem arises of providing suitable letters for the three component vectors. It would not do to call them **x**, **y**, and **z**, for these are displacements, or directed distances, and are measured in meters (abbreviated m) or some other unit of length. The problem is most often avoided by using *component scalars*, simply called *components*, F_x, F_y, and F_z. The components are the signed magnitudes of the component vectors. We may then write $\mathbf{F} = F_x\mathbf{a}_x + F_y\mathbf{a}_y + F_z\mathbf{a}_z$. The component vectors are $F_x\mathbf{a}_x$, $F_y\mathbf{a}_y$, and $F_z\mathbf{a}_z$.

Any vector **B** then may be described by $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The magnitude of **B** written $|\mathbf{B}|$ or simply B, is given by

$$\boxed{|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}} \tag{1}$$

Each of the three coordinate systems we discuss will have its three fundamental and mutually perpendicular unit vectors that are used to resolve any vector into its component vectors. Unit vectors are not limited to this application. It is helpful to write a unit vector having a specified direction. This is easily done, for a unit vector in a given direction is merely a vector in that direction divided by its magnitude. A unit vector in the **r** direction is $\mathbf{r}/\sqrt{x^2 + y^2 + z^2}$, and a unit vector in the direction of the vector **B** is

$$\boxed{\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}} \tag{2}$$

EXAMPLE 1.1

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$.

Solution. We first construct the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

We continue by finding the magnitude of **G**,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

and finally expressing the desired unit vector as the quotient,

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

A special symbol is desirable for a unit vector so that its character is immediately apparent. Symbols that have been used are $\mathbf{u}_B$, $\mathbf{a}_B$, $\mathbf{1}_\mathbf{B}$, or even **b**. We will consistently use the lowercase **a** with an appropriate subscript.

[NOTE: Throughout the text, drill problems appear following sections in which a new principle is introduced in order to allow students to test their understanding of the basic fact itself. The problems are useful in gaining familiarity with new terms and ideas and should all be worked. More general problems appear at the ends of the chapters. The answers to the drill problems are given in the same order as the parts of the problem.]

D1.1. Given points $M(-1, 2, 1)$, $N(3, -3, 0)$, and $P(-2, -3, -4)$, find: (*a*) $\mathbf{R}_{MN}$; (*b*) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (*c*) $|\mathbf{r}_M|$; (*d*) $\mathbf{a}_{MP}$; (*e*) $|2\mathbf{r}_P - 3\mathbf{r}_N|$.

Ans. $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$; $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$; 2.45; $-0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z$; 15.56

1.5 THE VECTOR FIELD

We have defined a vector field as a vector function of a position vector. In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question. Because we have considered only the rectangular coordinate system, we expect the vector to be a function of the variables x, y, and z.

If we again represent the position vector as $\mathbf{r}$, then a vector field $\mathbf{G}$ can be expressed in functional notation as $\mathbf{G}(\mathbf{r})$; a scalar field T is written as $T(\mathbf{r})$.

If we inspect the velocity of the water in the ocean in some region near the surface where tides and currents are important, we might decide to represent it by a velocity vector that is in any direction, even up or down. If the z axis is taken as upward, the x axis in a northerly direction, the y axis to the west, and the origin at the surface, we have a right-handed coordinate system and may write the velocity vector as $\mathbf{v} = v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z$, or $\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$; each of the components v_x, v_y, and v_z may be a function of the three variables x, y, and z. If we are in some portion of the Gulf Stream where the water is moving only to the north, then v_y and v_z are zero. Further simplifying assumptions might be made if the velocity falls off with depth and changes very slowly as we move north, south, east, or west. A suitable expression could be $\mathbf{v} = 2e^{z/100}\mathbf{a}_x$. We have a velocity of 2 m/s (meters per second) at the surface and a velocity of 0.368×2, or 0.736 m/s, at a depth of 100 m ($z = -100$). The velocity continues to decrease with depth, while maintaining a constant direction.

D1.2. A vector field $\mathbf{S}$ is expressed in rectangular coordinates as $\mathbf{S} = \{125/[(x-1)^2+(y-2)^2+(z+1)^2]\}\{(x-1)\mathbf{a}_x+(y-2)\mathbf{a}_y+(z+1)\mathbf{a}_z\}$. (*a*) Evaluate $\mathbf{S}$ at $P(2, 4, 3)$. (*b*) Determine a unit vector that gives the direction of $\mathbf{S}$ at P. (*c*) Specify the surface $f(x, y, z)$ on which $|\mathbf{S}| = 1$.

Ans. $5.95\mathbf{a}_x + 11.90\mathbf{a}_y + 23.8\mathbf{a}_z$; $0.218\mathbf{a}_x + 0.436\mathbf{a}_y + 0.873\mathbf{a}_z$; $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = 125$

1.6 THE DOT PRODUCT

We now consider the first of two types of vector multiplication. The second type will be discussed in the following section.

Given two vectors **A** and **B**, the *dot product*, or *scalar product*, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle between them,

$$\boxed{\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos \theta_{AB}} \tag{3}$$

The dot appears between the two vectors and should be made heavy for emphasis. The dot, or scalar, product is a scalar, as one of the names implies, and it obeys the commutative law,

$$\boxed{\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}} \tag{4}$$

for the sign of the angle does not affect the cosine term. The expression $\mathbf{A} \cdot \mathbf{B}$ is read "**A** dot **B**."

Perhaps the most common application of the dot product is in mechanics, where a constant force **F** applied over a straight displacement **L** does an amount of work $FL \cos \theta$, which is more easily written $\mathbf{F} \cdot \mathbf{L}$. We might anticipate one of the results of Chapter 4 by pointing out that if the force varies along the path, integration is necessary to find the total work, and the result becomes

$$\text{Work} = \int \mathbf{F} \cdot d\mathbf{L}$$

Another example might be taken from magnetic fields. The total flux Φ crossing a surface of area S is given by BS if the magnetic flux density B is perpendicular to the surface and uniform over it. We define a *vector surface* **S** as having area for its magnitude and having a direction *normal* to the surface (avoiding for the moment the problem of which of the two possible normals to take). The flux crossing the surface is then $\mathbf{B} \cdot \mathbf{S}$. This expression is valid for any direction of the uniform magnetic flux density. If the flux density is not constant over the surface, the total flux is $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$. Integrals of this general form appear in Chapter 3 when we study electric flux density.

Finding the angle between two vectors in three-dimensional space is often a job we would prefer to avoid, and for that reason the definition of the dot product is usually not used in its basic form. A more helpful result is obtained by considering two vectors whose rectangular components are given, such as $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ and $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The dot product also obeys the distributive law, and, therefore, $\mathbf{A} \cdot \mathbf{B}$ yields the sum of nine scalar terms, each involving the dot product of two unit vectors. Because the angle between two different unit vectors of the rectangular coordinate system is 90°, we then have

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_x = \mathbf{a}_x \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_y = 0$$

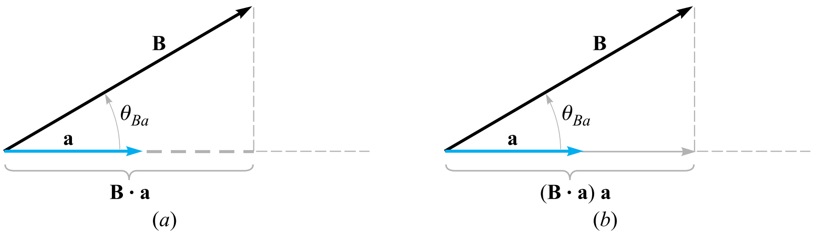

Figure 1.4 (*a*) The scalar component of **B** in the direction of the unit vector **a** is **B** · **a**. (*b*) The vector component of **B** in the direction of the unit vector **a** is (**B** · **a**)**a**.

The remaining three terms involve the dot product of a unit vector with itself, which is unity, giving finally

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z} \tag{5}$$

which is an expression involving no angles.

A vector dotted with itself yields the magnitude squared, or

$$\boxed{\mathbf{A}\cdot\mathbf{A} = A^2 = |\mathbf{A}|^2} \tag{6}$$

and any unit vector dotted with itself is unity,

$$\mathbf{a}_A \cdot \mathbf{a}_A = 1$$

One of the most important applications of the dot product is that of finding the component of a vector in a given direction. Referring to Figure 1.4*a*, we can obtain the component (scalar) of **B** in the direction specified by the unit vector **a** as

$$\mathbf{B}\cdot\mathbf{a} = |\mathbf{B}|\,|\mathbf{a}|\cos\theta_{Ba} = |\mathbf{B}|\cos\theta_{Ba}$$

The sign of the component is positive if $0 \le \theta_{Ba} \le 90^\circ$ and negative whenever $90^\circ \le \theta_{Ba} \le 180^\circ$.

To obtain the component *vector* of **B** in the direction of **a**, we multiply the component (scalar) by **a**, as illustrated by Figure 1.4*b*. For example, the component of **B** in the direction of $\mathbf{a}_x$ is $\mathbf{B}\cdot\mathbf{a}_x = B_x$, and the component vector is $B_x\mathbf{a}_x$, or $(\mathbf{B}\cdot\mathbf{a}_x)\mathbf{a}_x$. Hence, the problem of finding the component of a vector in any direction becomes the problem of finding a unit vector in that direction, and that we can do.

The geometrical term *projection* is also used with the dot product. Thus, **B** · **a** is the projection of **B** in the **a** direction.

EXAMPLE 1.2

In order to illustrate these definitions and operations, consider the vector field $\mathbf{G} = y\mathbf{a}_x - 2.5x\mathbf{a}_y + 3\mathbf{a}_z$ and the point $Q(4, 5, 2)$. We wish to find: **G** at Q; the scalar component of **G** at Q in the direction of $\mathbf{a}_N = \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$; the vector component of **G** at Q in the direction of $\mathbf{a}_N$; and finally, the angle θ_{Ga} between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$.

Solution. Substituting the coordinates of point Q into the expression for $\mathbf{G}$, we have

$$\mathbf{G}(\mathbf{r}_Q) = 5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z$$

Next we find the scalar component. Using the dot product, we have

$$\mathbf{G} \cdot \mathbf{a}_N = (5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z) \cdot \tfrac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = \tfrac{1}{3}(10 - 10 - 6) = -2$$

The vector component is obtained by multiplying the scalar component by the unit vector in the direction of $\mathbf{a}_N$,

$$(\mathbf{G} \cdot \mathbf{a}_N)\mathbf{a}_N = -(2)\tfrac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = -1.333\mathbf{a}_x - 0.667\mathbf{a}_y + 1.333\mathbf{a}_z$$

The angle between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$ is found from

$$\mathbf{G} \cdot \mathbf{a}_N = |\mathbf{G}| \cos\theta_{Ga}$$
$$-2 = \sqrt{25 + 100 + 9} \cos\theta_{Ga}$$

and

$$\theta_{Ga} = \cos^{-1} \frac{-2}{\sqrt{134}} = 99.9^\circ$$

D1.3. The three vertices of a triangle are located at $A(6, -1, 2)$, $B(-2, 3, -4)$, and $C(-3, 1, 5)$. Find: (*a*) $\mathbf{R}_{AB}$; (*b*) $\mathbf{R}_{AC}$; (*c*) the angle θ_{BAC} at vertex A; (*d*) the (vector) projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$.

Ans. $-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z$; $-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$; 53.6°; $-5.94\mathbf{a}_x + 1.319\mathbf{a}_y + 1.979\mathbf{a}_z$

1.7 THE CROSS PRODUCT

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, we now define the *cross product*, or *vector product*, of $\mathbf{A}$ and $\mathbf{B}$, written with a cross between the two vectors as $\mathbf{A} \times \mathbf{B}$ and read "$\mathbf{A}$ cross $\mathbf{B}$." The cross product $\mathbf{A} \times \mathbf{B}$ is a vector; the magnitude of $\mathbf{A} \times \mathbf{B}$ is equal to the product of the magnitudes of $\mathbf{A}$, $\mathbf{B}$, and the sine of the smaller angle between $\mathbf{A}$ and $\mathbf{B}$; the direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$ and is along one of the two possible perpendiculars which is in the direction of advance of a right-handed screw as $\mathbf{A}$ is turned into $\mathbf{B}$. This direction is illustrated in Figure 1.5. Remember that either vector may be moved about at will, maintaining its direction constant, until the two vectors have a "common origin." This determines the plane containing both. However, in most of our applications we will be concerned with vectors defined at the same point.

As an equation we can write

$$\boxed{\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\, |\mathbf{B}| \sin\theta_{AB}} \tag{7}$$

where an additional statement, such as that given above, is required to explain the direction of the unit vector $\mathbf{a}_N$. The subscript stands for "normal."

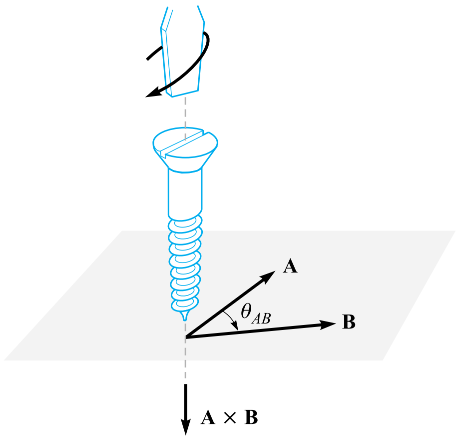

Figure 1.5 The direction of A × B is in the direction of advance of a right-handed screw as A is turned into B.

Reversing the order of the vectors **A** and **B** results in a unit vector in the opposite direction, and we see that the cross product is not commutative, for $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$. If the definition of the cross product is applied to the unit vectors $\mathbf{a}_x$ and $\mathbf{a}_y$, we find $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$, for each vector has unit magnitude, the two vectors are perpendicular, and the rotation of $\mathbf{a}_x$ into $\mathbf{a}_y$ indicates the positive z direction by the definition of a right-handed coordinate system. In a similar way, $\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$ and $\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$. Note the alphabetic symmetry. As long as the three vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are written in order (and assuming that $\mathbf{a}_x$ follows $\mathbf{a}_z$, like three elephants in a circle holding tails, so that we could also write $\mathbf{a}_y$, $\mathbf{a}_z$, $\mathbf{a}_x$ or $\mathbf{a}_z$, $\mathbf{a}_x$, $\mathbf{a}_y$), then the cross and equal sign may be placed in either of the two vacant spaces. As a matter of fact, it is now simpler to define a right-handed rectangular coordinate system by saying that $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$.

A simple example of the use of the cross product may be taken from geometry or trigonometry. To find the area of a parallelogram, the product of the lengths of two adjacent sides is multiplied by the sine of the angle between them. Using vector notation for the two sides, we then may express the (scalar) area as the *magnitude* of $\mathbf{A} \times \mathbf{B}$, or $|\mathbf{A} \times \mathbf{B}|$.

The cross product may be used to replace the right-hand rule familiar to all electrical engineers. Consider the force on a straight conductor of length **L**, where the direction assigned to **L** corresponds to the direction of the steady current I, and a uniform magnetic field of flux density **B** is present. Using vector notation, we may write the result neatly as $\mathbf{F} = I\mathbf{L} \times \mathbf{B}$. This relationship will be obtained later in Chapter 9.

The evaluation of a cross product by means of its definition turns out to be more work than the evaluation of the dot product from its definition, for not only must we find the angle between the vectors, but we must also find an expression for the unit vector $\mathbf{a}_N$. This work may be avoided by using rectangular components for the two vectors **A** and **B** and expanding the cross product as a sum of nine simpler cross products, each involving two unit vectors,

$$\begin{aligned}\mathbf{A}\times\mathbf{B} = {} & A_xB_x\mathbf{a}_x\times\mathbf{a}_x + A_xB_y\mathbf{a}_x\times\mathbf{a}_y + A_xB_z\mathbf{a}_x\times\mathbf{a}_z \\ & + A_yB_x\mathbf{a}_y\times\mathbf{a}_x + A_yB_y\mathbf{a}_y\times\mathbf{a}_y + A_yB_z\mathbf{a}_y\times\mathbf{a}_z \\ & + A_zB_x\mathbf{a}_z\times\mathbf{a}_x + A_zB_y\mathbf{a}_z\times\mathbf{a}_y + A_zB_z\mathbf{a}_z\times\mathbf{a}_z\end{aligned}$$

We have already found that $\mathbf{a}_x\times\mathbf{a}_y = \mathbf{a}_z$, $\mathbf{a}_y\times\mathbf{a}_z = \mathbf{a}_x$, and $\mathbf{a}_z\times\mathbf{a}_x = \mathbf{a}_y$. The three remaining terms are zero, for the cross product of any vector with itself is zero, since the included angle is zero. These results may be combined to give

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z \tag{8}$$

or written as a determinant in a more easily remembered form,

$$\boxed{\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{vmatrix}} \tag{9}$$

Thus, if $\mathbf{A} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = -4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$, we have

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ -4 & -2 & 5\end{vmatrix} \\ &= [(-3)(5) - (1(-2)]\mathbf{a}_x - [(2)(5) - (1)(-4)]\mathbf{a}_y + [(2)(-2) - (-3)(-4)]\mathbf{a}_z \\ &= -13\mathbf{a}_x - 14\mathbf{a}_y - 16\mathbf{a}_z\end{aligned}$$

D1.4. The three vertices of a triangle are located at $A(6, -1, 2)$, $B(-2, 3, -4)$, and $C(-3, 1, 5)$. Find: (*a*) $\mathbf{R}_{AB}\times\mathbf{R}_{AC}$; (*b*) the area of the triangle; (*c*) a unit vector perpendicular to the plane in which the triangle is located.

Ans. $24\mathbf{a}_x + 78\mathbf{a}_y + 20\mathbf{a}_z$; 42.0; $0.286\mathbf{a}_x + 0.928\mathbf{a}_y + 0.238\mathbf{a}_z$

1.8 OTHER COORDINATE SYSTEMS: CIRCULAR CYLINDRICAL COORDINATES

The rectangular coordinate system is generally the one in which students prefer to work every problem. This often means a lot more work, because many problems possess a type of symmetry that pleads for a more logical treatment. It is easier to do now, once and for all, the work required to become familiar with cylindrical and spherical coordinates, instead of applying an equal or greater effort to every problem involving cylindrical or spherical symmetry later. With this in mind, we will take a careful and unhurried look at cylindrical and spherical coordinates.

The circular cylindrical coordinate system is the three-dimensional version of the polar coordinates of analytic geometry. In polar coordinates, a point is located in a plane by giving both its distance ρ from the origin and the angle ϕ between the line from the point to the origin and an arbitrary radial line, taken as $\phi = 0$.[4] In circular cylindrical coordinates, we also specify the distance z of the point from an arbitrary $z = 0$ reference plane that is perpendicular to the line $\rho = 0$. For simplicity, we usually refer to circular cylindrical coordinates simply as cylindrical coordinates. This will not cause any confusion in reading this book, but it is only fair to point out that there are such systems as elliptic cylindrical coordinates, hyperbolic cylindrical coordinates, parabolic cylindrical coordinates, and others.

We no longer set up three axes as with rectangular coordinates, but we must instead consider any point as the intersection of three mutually perpendicular surfaces. These surfaces are a circular cylinder ($\rho =$ constant), a plane ($\phi =$ constant), and another plane ($z =$ constant). This corresponds to the location of a point in a rectangular coordinate system by the intersection of three planes ($x =$ constant, $y =$ constant, and $z =$ constant). The three surfaces of circular cylindrical coordinates are shown in Figure 1.6*a*. Note that three such surfaces may be passed through any point, unless it lies on the z axis, in which case one plane suffices.

Three unit vectors must also be defined, but we may no longer direct them along the "coordinate axes," for such axes exist only in rectangular coordinates. Instead, we take a broader view of the unit vectors in rectangular coordinates and realize that they are directed toward increasing coordinate values and are perpendicular to the surface on which that coordinate value is constant (i.e., the unit vector $\mathbf{a}_x$ is normal to the plane $x =$ constant and points toward larger values of x). In a corresponding way we may now define three unit vectors in cylindrical coordinates, $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$.

The unit vector $\mathbf{a}_\rho$ at a point $P(\rho_1, \phi_1, z_1)$ is directed radially outward, normal to the cylindrical surface $\rho = \rho_1$. It lies in the planes $\phi = \phi_1$ and $z = z_1$. The unit vector $\mathbf{a}_\phi$ is normal to the plane $\phi = \phi_1$, points in the direction of increasing ϕ, lies in the plane $z = z_1$, and is tangent to the cylindrical surface $\rho = \rho_1$. The unit vector $\mathbf{a}_z$ is the same as the unit vector $\mathbf{a}_z$ of the rectangular coordinate system. Figure 1.6*b* shows the three vectors in cylindrical coordinates.

In rectangular coordinates, the unit vectors are not functions of the coordinates. Two of the unit vectors in cylindrical coordinates, $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$, however, *do* vary with the coordinate ϕ, as their directions change. In integration or differentiation with respect to ϕ, then, $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$ must not be treated as constants.

The unit vectors are again mutually perpendicular, for each is normal to one of the three mutually perpendicular surfaces, and we may define a right-handed cylindrical

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[4] The two variables of polar coordinates are commonly called r and θ. With three coordinates, however, it is more common to use ρ for the radius variable of cylindrical coordinates and r for the (different) radius variable of spherical coordinates. Also, the angle variable of cylindrical coordinates is customarily called ϕ because everyone uses θ for a different angle in spherical coordinates. The angle ϕ is common to both cylindrical and spherical coordinates. See?

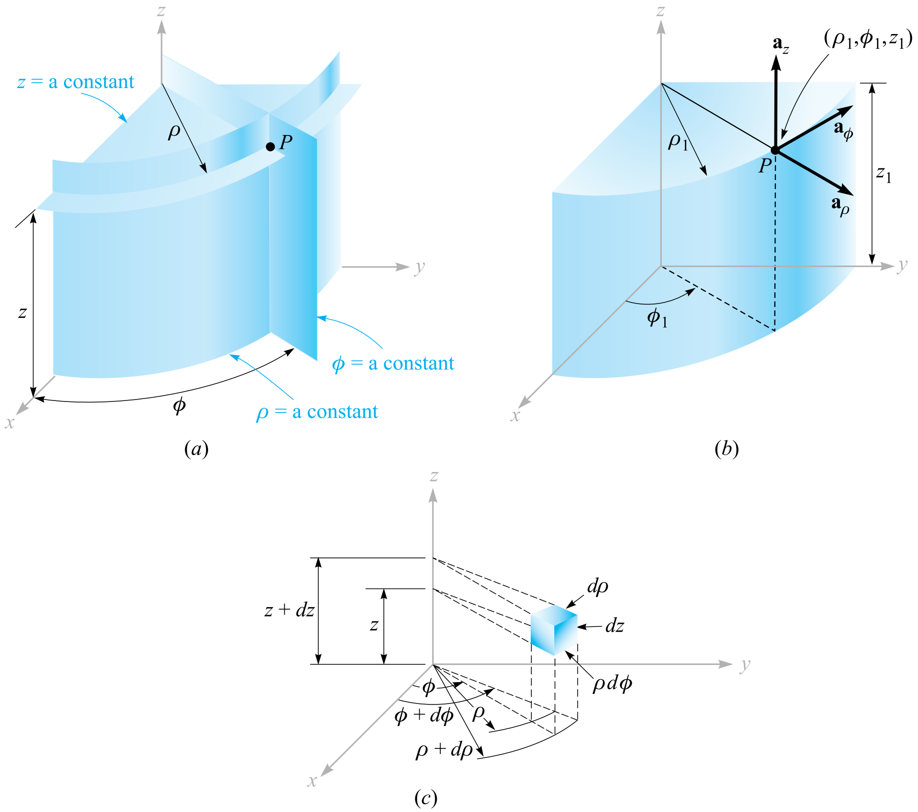

Figure 1.6 (*a*) The three mutually perpendicular surfaces of the circular cylindrical coordinate system. (*b*) The three unit vectors of the circular cylindrical coordinate system. (*c*) The differential volume unit in the circular cylindrical coordinate system; $d\rho$, $\rho d\phi$, and dz are all elements of length.

coordinate system as one in which $\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$, or (for those who have flexible fingers) as one in which the thumb, forefinger, and middle finger point in the direction of increasing ρ, ϕ, and z, respectively.

A differential volume element in cylindrical coordinates may be obtained by increasing ρ, ϕ, and z by the differential increments $d\rho$, $d\phi$, and dz. The two cylinders of radius ρ and $\rho + d\rho$, the two radial planes at angles ϕ and $\phi + d\phi$, and the two "horizontal" planes at "elevations" z and $z + dz$ now enclose a small volume, as shown in Figure 1.6*c*, having the shape of a truncated wedge. As the volume element becomes very small, its shape approaches that of a rectangular parallelepiped having sides of length $d\rho$, $\rho d\phi$, and dz. Note that $d\rho$ and dz are dimensionally lengths, but $d\phi$ is not; $\rho d\phi$ is the length. The surfaces have areas of $\rho\, d\rho\, d\phi$, $d\rho\, dz$, and $\rho\, d\phi\, dz$, and the volume becomes $\rho\, d\rho\, d\phi\, dz$.

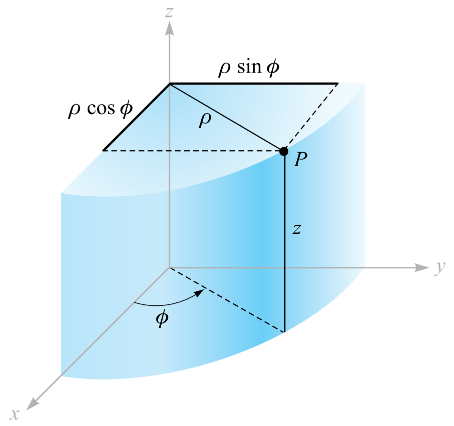

Figure 1.7 The relationship between the rectangular variables x, y, z and the cylindrical coordinate variables ρ, ϕ, z. There is no change in the variable z between the two systems.

The variables of the rectangular and cylindrical coordinate systems are easily related to each other. Referring to Figure 1.7, we see that

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned} \tag{10}$$

From the other viewpoint, we may express the cylindrical variables in terms of x, y, and z:

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \quad (\rho \geq 0) \\ \phi &= \tan^{-1} \frac{y}{x} \\ z &= z \end{aligned} \tag{11}$$

We consider the variable ρ to be positive or zero, thus using only the positive sign for the radical in (11). The proper value of the angle ϕ is determined by inspecting the signs of x and y. Thus, if $x = -3$ and $y = 4$, we find that the point lies in the second quadrant so that $\rho = 5$ and $\phi = 126.9^\circ$. For $x = 3$ and $y = -4$, we have $\phi = -53.1^\circ$ or 306.9°, whichever is more convenient.

Using (10) or (11), scalar functions given in one coordinate system are easily transformed into the other system.

A vector function in one coordinate system, however, requires two steps in order to transform it to another coordinate system, because a different set of component vectors is generally required. That is, we may be given a rectangular vector

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

where each component is given as a function of x, y, and z, and we need a vector in cylindrical coordinates

$$\mathbf{A} = A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$

where each component is given as a function of ρ, ϕ, and z.

To find any desired component of a vector, we recall from the discussion of the dot product that a component in a desired direction may be obtained by taking the dot product of the vector and a unit vector in the desired direction. Hence,

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho \qquad \text{and} \qquad A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi$$

Expanding these dot products, we have

$$A_\rho = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\rho = A_x\mathbf{a}_x\cdot\mathbf{a}_\rho + A_y\mathbf{a}_y\cdot\mathbf{a}_\rho \tag{12}$$

$$A_\phi = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\phi = A_x\mathbf{a}_x\cdot\mathbf{a}_\phi + A_y\mathbf{a}_y\cdot\mathbf{a}_\phi \tag{13}$$

and

$$A_z = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_z = A_z\mathbf{a}_z\cdot\mathbf{a}_z = A_z \tag{14}$$

since $\mathbf{a}_z\cdot\mathbf{a}_\rho$ and $\mathbf{a}_z\cdot\mathbf{a}_\phi$ are zero.

In order to complete the transformation of the components, it is necessary to know the dot products $\mathbf{a}_x\cdot\mathbf{a}_\rho$, $\mathbf{a}_y\cdot\mathbf{a}_\rho$, $\mathbf{a}_x\cdot\mathbf{a}_\phi$, and $\mathbf{a}_y\cdot\mathbf{a}_\phi$. Applying the definition of the dot product, we see that since we are concerned with unit vectors, the result is merely the cosine of the angle between the two unit vectors in question. Referring to Figure 1.7 and thinking mightily, we identify the angle between $\mathbf{a}_x$ and $\mathbf{a}_\rho$ as ϕ, and thus $\mathbf{a}_x\cdot\mathbf{a}_\rho = \cos\phi$, but the angle between $\mathbf{a}_y$ and $\mathbf{a}_\rho$ is $90^\circ - \phi$, and $\mathbf{a}_y\cdot\mathbf{a}_\rho = \cos(90^\circ - \phi) = \sin\phi$. The remaining dot products of the unit vectors are found in a similar manner, and the results are tabulated as functions of ϕ in Table 1.1.

Transforming vectors from rectangular to cylindrical coordinates or vice versa is therefore accomplished by using (10) or (11) to change variables, and by using the dot products of the unit vectors given in Table 1.1 to change components. The two steps may be taken in either order.

Table 1.1 Dot products of unit vectors in cylindrical and rectangular coordinate systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	1

EXAMPLE 1.3

Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

Solution. The new components are

$$\begin{aligned}
B_\rho &= \mathbf{B}\cdot\mathbf{a}_\rho = y(\mathbf{a}_x\cdot\mathbf{a}_\rho) - x(\mathbf{a}_y\cdot\mathbf{a}_\rho)\\
&= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0\\
B_\phi &= \mathbf{B}\cdot\mathbf{a}_\phi = y(\mathbf{a}_x\cdot\mathbf{a}_\phi) - x(\mathbf{a}_y\cdot\mathbf{a}_\phi)\\
&= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho
\end{aligned}$$

Thus,

$$\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$$

D1.5. (a) Give the rectangular coordinates of the point $C(\rho = 4.4, \phi = -115^\circ, z = 2)$. ($b$) Give the cylindrical coordinates of the point $D(x = -3.1, y = 2.6, z = -3)$. ($c$) Specify the distance from C to D.

Ans. $C(x = -1.860, y = -3.99, z = 2)$; $D(\rho = 4.05, \phi = 140.0^\circ, z = -3)$; 8.36

D1.6. Transform to cylindrical coordinates: (a) $\mathbf{F} = 10\mathbf{a}_x - 8\mathbf{a}_y + 6\mathbf{a}_z$ at point $P(10, -8, 6)$; (b) $\mathbf{G} = (2x + y)\mathbf{a}_x - (y - 4x)\mathbf{a}_y$ at point $Q(\rho, \phi, z)$. (c) Give the rectangular components of the vector $\mathbf{H} = 20\mathbf{a}_\rho - 10\mathbf{a}_\phi + 3\mathbf{a}_z$ at $P(x = 5, y = 2, z = -1)$.

Ans. $12.81\mathbf{a}_\rho + 6\mathbf{a}_z$; $(2\rho\cos^2\phi - \rho\sin^2\phi + 5\rho\sin\phi\cos\phi)\mathbf{a}_\rho + (4\rho\cos^2\phi - \rho\sin^2\phi - 3\rho\sin\phi\cos\phi)\mathbf{a}_\phi$; $H_x = 22.3$, $H_y = -1.857$, $H_z = 3$

1.9 THE SPHERICAL COORDINATE SYSTEM

We have no two-dimensional coordinate system to help us understand the three-dimensional spherical coordinate system, as we have for the circular cylindrical coordinate system. In certain respects we can draw on our knowledge of the latitude-and-longitude system of locating a place on the surface of the earth, but usually we consider only points on the surface and not those below or above ground.

Let us start by building a spherical coordinate system on the three rectangular axes (Figure 1.8a). We first define the distance from the origin to any point as r. The surface r = constant is a sphere.

The second coordinate is an angle θ between the z axis and the line drawn from the origin to the point in question. The surface θ = constant is a cone, and the two surfaces, cone and sphere, are everywhere perpendicular along their intersection, which is a circle of radius $r\sin\theta$. The coordinate θ corresponds to latitude,

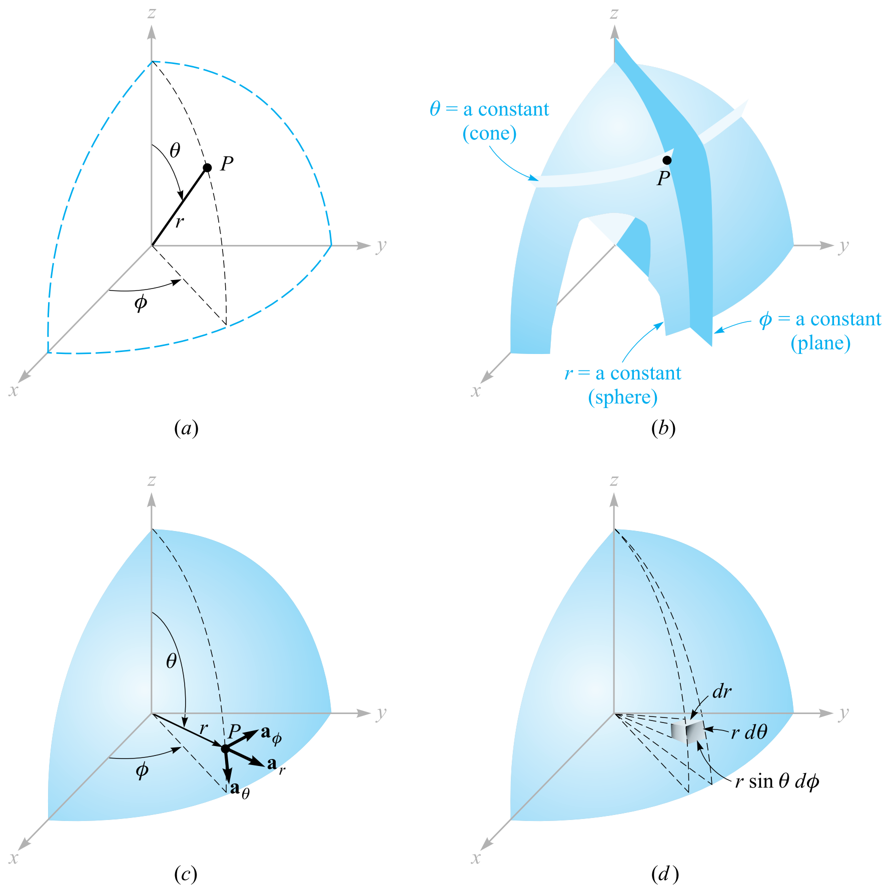

Figure 1.8 (*a*) The three spherical coordinates. (*b*) The three mutually perpendicular surfaces of the spherical coordinate system. (*c*) The three unit vectors of spherical coordinates: $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$. (*d*) The differential volume element in the spherical coordinate system.

except that latitude is measured from the equator and θ is measured from the "North Pole."

The third coordinate ϕ is also an angle and is exactly the same as the angle ϕ of cylindrical coordinates. It is the angle between the x axis and the projection in the $z = 0$ plane of the line drawn from the origin to the point. It corresponds to the angle of longitude, but the angle ϕ increases to the "east." The surface $\phi =$ constant is a plane passing through the $\theta = 0$ line (or the z axis).

We again consider any point as the intersection of three mutually perpendicular surfaces—a sphere, a cone, and a plane—each oriented in the manner just described. The three surfaces are shown in Figure 1.8*b*.

Three unit vectors may again be defined at any point. Each unit vector is perpendicular to one of the three mutually perpendicular surfaces and oriented in that direction in which the coordinate increases. The unit vector $\mathbf{a}_r$ is directed radially outward, normal to the sphere $r =$ constant, and lies in the cone $\theta =$ constant and the plane $\phi =$ constant. The unit vector $\mathbf{a}_\theta$ is normal to the conical surface, lies in the plane, and is tangent to the sphere. It is directed along a line of "longitude" and points "south." The third unit vector $\mathbf{a}_\phi$ is the same as in cylindrical coordinates, being normal to the plane and tangent to both the cone and the sphere. It is directed to the "east."

The three unit vectors are shown in Figure 1.8c. They are, of course, mutually perpendicular, and a right-handed coordinate system is defined by causing $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$. Our system is right-handed, as an inspection of Figure 1.8c will show, on application of the definition of the cross product. The right-hand rule identifies the thumb, forefinger, and middle finger with the direction of increasing r, θ, and ϕ, respectively. (Note that the identification in cylindrical coordinates was with ρ, ϕ, and z, and in rectangular coordinates with x, y, and z.) A differential volume element may be constructed in spherical coordinates by increasing r, θ, and ϕ by dr, $d\theta$, and $d\phi$, as shown in Figure 1.8d. The distance between the two spherical surfaces of radius r and $r + dr$ is dr; the distance between the two cones having generating angles of θ and $\theta + d\theta$ is $r d\theta$; and the distance between the two radial planes at angles ϕ and $\phi + d\phi$ is found to be $r \sin\theta d\phi$, after a few moments of trigonometric thought. The surfaces have areas of $r\,dr\,d\theta$, $r \sin\theta\,dr\,d\phi$, and $r^2 \sin\theta\,d\theta\,d\phi$, and the volume is $r^2 \sin\theta\,dr\,d\theta\,d\phi$.

The transformation of scalars from the rectangular to the spherical coordinate system is easily made by using Figure 1.8a to relate the two sets of variables:

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{15}$$

The transformation in the reverse direction is achieved with the help of

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} && (r \geq 0) \\ \theta &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} && (0^\circ \leq \theta \leq 180^\circ) \\ \phi &= \tan^{-1} \frac{y}{x} \end{aligned} \tag{16}$$

The radius variable r is nonnegative, and θ is restricted to the range from 0° to 180°, inclusive. The angles are placed in the proper quadrants by inspecting the signs of x, y, and z.

The transformation of vectors requires us to determine the products of the unit vectors in rectangular and spherical coordinates. We work out these products from Figure 1.8c and a pinch of trigonometry. Because the dot product of any spherical unit vector with any rectangular unit vector is the component of the spherical

Table 1.2 Dot products of unit vectors in spherical and rectangular coordinate systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x\cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y\cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z\cdot$	$\cos\theta$	$-\sin\theta$	0

vector in the direction of the rectangular vector, the dot products with $\mathbf{a}_z$ are found to be

$$\mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$$
$$\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin\theta$$
$$\mathbf{a}_z \cdot \mathbf{a}_\phi = 0$$

The dot products involving $\mathbf{a}_x$ and $\mathbf{a}_y$ require first the projection of the spherical unit vector on the xy plane and then the projection onto the desired axis. For example, $\mathbf{a}_r \cdot \mathbf{a}_x$ is obtained by projecting $\mathbf{a}_r$ onto the xy plane, giving $\sin\theta$, and then projecting $\sin\theta$ on the x axis, which yields $\sin\theta\cos\phi$. The other dot products are found in a like manner, and all are shown in Table 1.2.

EXAMPLE 1.4

We illustrate this procedure by transforming the vector field $\mathbf{G} = (xz/y)\mathbf{a}_x$ into spherical components and variables.

Solution. We find the three spherical components by dotting $\mathbf{G}$ with the appropriate unit vectors, and we change variables during the procedure:

$$G_r = \mathbf{G}\cdot\mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi$$
$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$
$$G_\theta = \mathbf{G}\cdot\mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi$$
$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$
$$G\phi = \mathbf{G}\cdot\mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi)$$
$$= -r\cos\theta\cos\phi$$

Collecting these results, we have

$$\mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$

Appendix A describes the general curvilinear coordinate system of which the rectangular, circular cylindrical, and spherical coordinate systems are special cases. The first section of this appendix could well be scanned now.

D1.7. Given the two points, $C(-3, 2, 1)$ and $D(r = 5, \theta = 20^\circ, \phi = -70^\circ)$, find: (*a*) the spherical coordinates of C; (*b*) the rectangular coordinates of D; (*c*) the distance from C to D.

Ans. $C(r = 3.74, \theta = 74.5^\circ, \phi = 146.3^\circ)$; $D(x = 0.585, y = -1.607, z = 4.70)$; 6.29

D1.8. Transform the following vectors to spherical coordinates at the points given: (*a*) $10\mathbf{a}_x$ at $P(x = -3, y = 2, z = 4)$; (*b*) $10\mathbf{a}_y$ at $Q(\rho = 5, \phi = 30^\circ, z = 4)$; (*c*) $10\mathbf{a}_z$ at $M(r = 4, \theta = 110^\circ, \phi = 120^\circ)$.

Ans. $-5.57\mathbf{a}_r - 6.18\mathbf{a}_\theta - 5.55\mathbf{a}_\phi$; $3.90\mathbf{a}_r + 3.12\mathbf{a}_\theta + 8.66\mathbf{a}_\phi$; $-3.42\mathbf{a}_r - 9.40\mathbf{a}_\theta$

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CHAPTER 1 PROBLEMS

WWW Quizzes

1.1 Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find: (*a*) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$; (*b*) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$; (*c*) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$.

1.2 Vector **A** extends from the origin to (1, 2, 3), and vector **B** extends from the origin to (2, 3, −2). Find (*a*) the unit vector in the direction of $(\mathbf{A} - \mathbf{B})$; (*b*) the unit vector in the direction of the line extending from the origin to the midpoint of the line joining the ends of **A** and **B**.

1.3 The vector from the origin to point A is given as (6, −2, −4), and the unit vector directed from the origin toward point B is (2, −2, 1)/3. If points A and B are ten units apart, find the coordinates of point B.

1.4 A circle, centered at the origin with a radius of 2 units, lies in the xy plane. Determine the unit vector in rectangular components that lies in the xy plane, is tangent to the circle at $(-\sqrt{3}, 1, 0)$, and is in the general direction of increasing values of y.

1.5 A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2 + 2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find (a) $\mathbf{G}$ at P; (b) a unit vector in the direction of $\mathbf{G}$ at Q; (c) a unit vector directed from Q toward P; (d) the equation of the surface on which $|\mathbf{G}| = 60$.

1.6 Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of (a) the dot product; (b) the cross product.

1.7 Given the vector field $\mathbf{E} = 4zy^2 \cos 2x\mathbf{a}_x + 2zy \sin 2x\mathbf{a}_y + y^2 \sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find (a) the surfaces on which $E_y = 0$; (b) the region in which $E_y = E_z$; (c) the region in which $\mathbf{E} = 0$.

1.8 Demonstrate the ambiguity that results when the cross product is used to find the angle between two vectors by finding the angle between $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$. Does this ambiguity exist when the dot product is used?

1.9 A field is given as $\mathbf{G} = [25/(x^2 + y^2)](x\mathbf{a}_x + y\mathbf{a}_y)$. Find (a) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$; (b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P; (c) the value of the following double integral on the plane $y = 7$.

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y \, dz dx$$

1.10 By expressing diagonals as vectors and using the definition of the dot product, find the smaller angle between any two diagonals of a cube, where each diagonal connects diametrically opposite corners and passes through the center of the cube.

1.11 Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find (a) the vector $\mathbf{R}_{MN}$; (b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$; (c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$; (d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$.

1.12 Write an expression in rectangular components for the vector that extends from (x_1, y_1, z_1) to (x_2, y_2, z_2) and determine the magnitude of this vector.

1.13 Find (a) the vector component of $\mathbf{F} = 10\mathbf{a}_x - 6\mathbf{a}_y + 5\mathbf{a}_z$ that is parallel to $\mathbf{G} = 0.1\mathbf{a}_x + 0.2\mathbf{a}_y + 0.3\mathbf{a}_z$; (b) the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$; (c) the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$.

1.14 Given that $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$, where the three vectors represent line segments and extend from a common origin, must the three vectors be coplanar? If $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0$, are the four vectors coplanar?

1.15 Three vectors extending from the origin are given as $\mathbf{r}_1 = (7, 3, -2)$, $\mathbf{r}_2 = (-2, 7, -3)$, and $\mathbf{r}_3 = (0, 2, 3)$. Find (*a*) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$; (*b*) a unit vector perpendicular to the vectors $\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{r}_3$; (*c*) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$; (*d*) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

1.16 If $\mathbf{A}$ represents a vector one unit long directed due east, $\mathbf{B}$ represents a vector three units long directed due north, and $\mathbf{A} + \mathbf{B} = 2\mathbf{C} - \mathbf{D}$ and $2\mathbf{A} - \mathbf{B} = \mathbf{C} + 2\mathbf{D}$, determine the length and direction of $\mathbf{C}$.

1.17 Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18 - 10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle. Find (*a*) a unit vector perpendicular to the triangle; (*b*) a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$; (*c*) a unit vector in the plane of the triangle that bisects the interior angle at A.

1.18 A certain vector field is given as $\mathbf{G} = (y + 1)\mathbf{a}_x + x\mathbf{a}_y$. (*a*) Determine $\mathbf{G}$ at the point $(3, -2, 4)$; (*b*) obtain a unit vector defining the direction of $\mathbf{G}$ at $(3, -2, 4)$.

1.19 (*a*) Express the field $\mathbf{D} = (x^2 + y^2)^{-1}(x\mathbf{a}_x + y\mathbf{a}_y)$ in cylindrical components and cylindrical variables. (*b*) Evaluate $\mathbf{D}$ at the point where $\rho = 2$, $\phi = 0.2\pi$, and $z = 5$, expressing the result in cylindrical and rectangular components.

1.20 If the three sides of a triangle are represented by vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, all directed counterclockwise, show that $|\mathbf{C}|^2 = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$ and expand the product to obtain the law of cosines.

1.21 Express in cylindrical components: (*a*) the vector from $C(3, 2, -7)$ to $D(-1, -4, 2)$; (*b*) a unit vector at D directed toward C; (*c*) a unit vector at D directed toward the origin.

1.22 A sphere of radius a, centered at the origin, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is clockwise when one is looking in the positive z direction. (*a*) Using spherical components, write an expression for the velocity field, $\mathbf{v}$, that gives the tangential velocity at any point within the sphere; (*b*) convert to rectangular components.

1.23 The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100^\circ$, $\phi = 130^\circ$, $z = 3$, and $z = 4.5$ define a closed surface. Find (*a*) the enclosed volume; (*b*) the total area of the enclosing surface; (*c*) the total length of the twelve edges of the surfaces; (*d*) the length of the longest straight line that lies entirely within the volume.

1.24 Two unit vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$, lie in the xy plane and pass through the origin. They make angles ϕ_1 and ϕ_2, respectively, with the x axis (*a*) Express each vector in rectangular components; (*b*) take the dot product and verify the trigonometric identity, $\cos(\phi_1 - \phi_2) = \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2$; (*c*) take the cross product and verify the trigonometric identity $\sin(\phi_2 - \phi_1) = \sin\phi_2 \cos\phi_1 - \cos\phi_2 \sin\phi_1$.

1.25 Given point $P(r = 0.8, \theta = 30°, \phi = 45°)$ and $\mathbf{E} = 1/r^2\ [\cos\phi\, \mathbf{a}_r + (\sin\phi/\sin\theta)\, \mathbf{a}_\phi]$, find (*a*) $\mathbf{E}$ at P; (*b*) $|\mathbf{E}|$ at P; (*c*) a unit vector in the direction of $\mathbf{E}$ at P.

1.26 Express the uniform vector field $\mathbf{F} = 5\mathbf{a}_x$ in (*a*) cylindrical components; (*b*) spherical components.

1.27 The surfaces $r = 2$ and 4, $\theta = 30°$ and $50°$, and $\phi = 20°$ and $60°$ identify a closed surface. Find (*a*) the enclosed volume; (*b*) the total area of the enclosing surface; (*c*) the total length of the twelve edges of the surface; (*d*) the length of the longest straight line that lies entirely within the surface.

1.28 State whether or not $\mathbf{A} = \mathbf{B}$ and, if not, what conditions are imposed on $\mathbf{A}$ and $\mathbf{B}$ when (*a*) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$; (*b*) $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$; (*c*) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$ *and* $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$; (*d*) $\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$ *and* $\mathbf{A} \times \mathbf{C} = \mathbf{B} \times \mathbf{C}$ where $\mathbf{C}$ is any vector except $\mathbf{C} = 0$.

1.29 Express the unit vector $\mathbf{a}_x$ in spherical components at the point: (*a*) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad; (*b*) $x = 3$, $y = 2$, $z = -1$; (*c*) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$.

1.30 Consider a problem analogous to the varying wind velocities encountered by transcontinental aircraft. We assume a constant altitude, a plane earth, a flight along the x axis from 0 to 10 units, no vertical velocity component, and no change in wind velocity with time. Assume $\mathbf{a}_x$ to be directed to the east and $\mathbf{a}_y$ to the north. The wind velocity at the operating altitude is assumed to be:

$$\mathbf{v}(x, y) = \frac{(0.01x^2 - 0.08x + 0.66)\mathbf{a}_x - (0.05x - 0.4)\mathbf{a}_y}{1 + 0.5y^2}$$

Determine the location and magnitude of (*a*) the maximum tailwind encountered; (*b*) repeat for headwind; (*c*) repeat for crosswind; (*d*) Would more favorable tailwinds be available at some other latitude? If so, where?

2 CHAPTER

Coulomb's Law and Electric Field Intensity

Having formulated the language of vector analysis in the first chapter, we next establish and describe a few basic principles of electricity. In this chapter, we introduce Coulomb's electrostatic force law and then formulate this in a general way using field theory. The tools that will be developed can be used to solve any problem in which forces between static charges are to be evaluated or to determine the electric field that is associated with any charge distribution. Initially, we will restrict the study to fields in *vacuum* or *free space*; this would apply to media such as air and other gases. Other materials are introduced in Chapters 5 and 6 and time-varying fields are introduced in Chapter 9. ■

2.1 THE EXPERIMENTAL LAW OF COULOMB

Records from at least 600 B.C. show evidence of the knowledge of static electricity. The Greeks were responsible for the term *electricity*, derived from their word for amber, and they spent many leisure hours rubbing a small piece of amber on their sleeves and observing how it would then attract pieces of fluff and stuff. However, their main interest lay in philosophy and logic, not in experimental science, and it was many centuries before the attracting effect was considered to be anything other than magic or a "life force."

Dr. Gilbert, physician to Her Majesty the Queen of England, was the first to do any true experimental work with this effect, and in 1600 he stated that glass, sulfur, amber, and other materials, which he named, would "not only draw to themselves straws and chaff, but all metals, wood, leaves, stone, earths, even water and oil."

Shortly thereafter, an officer in the French Army Engineers, Colonel Charles Coulomb, performed an elaborate series of experiments using a delicate torsion balance, invented by himself, to determine quantitatively the force exerted between two objects, each having a static charge of electricity. His published result bears a great similarity to Newton's gravitational law (discovered about a hundred years earlier).

Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance, which is large compared to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = k\frac{Q_1 Q_2}{R^2}$$

where Q_1 and Q_2 are the positive or negative quantities of charge, R is the separation, and k is a proportionality constant. If the International System of Units[1] (SI) is used, Q is measured in coulombs (C), R is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality k is written as

$$k = \frac{1}{4\pi\epsilon_0}$$

The new constant ϵ_0 is called the *permittivity of free space* and has magnitude, measured in farads per meter (F/m),

$$\boxed{\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi} 10^{-9} \text{ F/m}} \tag{1}$$

The quantity ϵ_0 is not dimensionless, for Coulomb's law shows that it has the label $C^2/N \cdot m^2$. We will later define the farad and show that it has the dimensions $C^2/N \cdot m$; we have anticipated this definition by using the unit F/m in equation (1).

Coulomb's law is now

$$\boxed{F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}} \tag{2}$$

The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in SI units as 1.602×10^{-19} C; hence a negative charge of one coulomb represents about 6×10^{18} electrons.[2] Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is 9×10^9 N, or about one million tons. The electron has a rest mass of 9.109×10^{-31}kg and has a radius of the order of magnitude of 3.8×10^{-15} m. This does not mean that the electron is spherical in shape, but merely describes the size of the region in which a slowly moving electron has the greatest probability of being found. All other known charged particles, including the proton, have larger masses and larger radii, and occupy a probabilistic volume larger than does the electron.

In order to write the vector form of (2), we need the additional fact (furnished also by Colonel Coulomb) that the force acts along the line joining the two charges

[1] The International System of Units (an mks system) is described in Appendix B. Abbreviations for the units are given in Table B.1. Conversions to other systems of units are given in Table B.2, while the prefixes designating powers of ten in SI appear in Table B.3.

[2] The charge and mass of an electron and other physical constants are tabulated in Table C.4 of Appendix C.

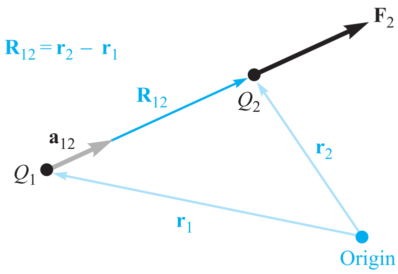

Figure 2.1 If Q_1 and Q_2 have like signs, the vector force $\mathbf{F}_2$ on Q_2 is in the same direction as the vector $\mathbf{R}_{12}$.

and is repulsive if the charges are alike in sign or attractive if they are of opposite sign. Let the vector $\mathbf{r}_1$ locate Q_1, whereas $\mathbf{r}_2$ locates Q_2. Then the vector $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ represents the directed line segment from Q_1 to Q_2, as shown in Figure 2.1. The vector $\mathbf{F}_2$ is the force on Q_2 and is shown for the case where Q_1 and Q_2 have the same sign. The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{3}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of R_{12}, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4}$$

EXAMPLE 2.1

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5)$ in a vacuum. We desire the force exerted on Q_2 by Q_1.

Solution. We use (3) and (4) to obtain the vector force. The vector $\mathbf{R}_{12}$ is

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\begin{aligned}\mathbf{F}_2 &= \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)\\ &= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{ N}\end{aligned}$$

The magnitude of the force is 30 N, and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on Q_2 may also be considered as three component forces,

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction. We might equally well have written

$$\boxed{\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}} \tag{5}$$

Coulomb's law is linear, for if we multiply Q_1 by a factor n, the force on Q_2 is also multiplied by the same factor n. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

D2.1. A charge $Q_A = -20\,\mu\text{C}$ is located at $A(-6, 4, 7)$, and a charge $Q_B = 50\,\mu\text{C}$ is at $B(5, 8, -2)$ in free space. If distances are given in meters, find: (*a*) $\mathbf{R}_{AB}$; (*b*) R_{AB}. Determine the vector force exerted on Q_A by Q_B if $\epsilon_0 =$ (*c*) $10^{-9}/(36\pi)$ F/m; (*d*) 8.854×10^{-12} F/m.

Ans. $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; 14.76 m; $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$ mN; $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$ mN

2.2 ELECTRIC FIELD INTENSITY

If we now consider one charge fixed in position, say Q_1, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force *field* that is associated with charge, Q_1. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives the *electric field intensity*, $\mathbf{E}_1$ arising from Q_1:

$$\mathbf{E}_1 = \frac{\mathbf{F}_t}{Q_1} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t} \tag{6}$$

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$\mathbf{E}_1$ is interpreted as the vector force, arising from charge Q_1, that acts on a unit positive test charge. More generally, we write the defining expression:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}} \tag{7}$$

in which $\mathbf{E}$, a vector function, is the electric field intensity *evaluated at the test charge location* that arises from all *other* charges in the vicinity—meaning the electric field arising from the test charge itself is not included in $\mathbf{E}$.

The units of $\mathbf{E}$ would be in force per unit charge (newtons per coulomb). Again anticipating a new dimensional quantity, the *volt* (V), having the label of joules per coulomb (J/C), or newton-meters per coulomb ($\mathrm{N \cdot m/C}$), we measure electric field intensity in the practical units of volts per meter (V/m).

Now, we dispense with most of the subscripts in (6), reserving the right to use them again any time there is a possibility of misunderstanding. The electric field of a single point charge becomes:

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R} \tag{8}$$

We remember that R is the magnitude of the vector $\mathbf{R}$, the directed line segment from the point at which the point charge Q is located to the point at which $\mathbf{E}$ is desired, and $\mathbf{a}_R$ is a unit vector in the $\mathbf{R}$ direction.[3]

We arbitrarily locate Q_1 at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and R is r. Hence

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\mathbf{a}_r \tag{9}$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

If we consider a charge that is *not* at the origin of our coordinate system, the field no longer possesses spherical symmetry, and we might as well use rectangular coordinates. For a charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$, as illustrated in Figure 2.2, we find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ by expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$, and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$
$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \tag{10}$$

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting $\mathbf{E}$ be symbolized in functional notation by $\mathbf{E}(\mathbf{r})$.

Because the coulomb forces are linear, the electric field intensity arising from two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of $(\mathbf{r}-\mathbf{r}_1)$ and $(\mathbf{r}-\mathbf{r}_2)$, respectively. The vectors $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}-\mathbf{r}_1$, $\mathbf{r}-\mathbf{r}_2$, $\mathbf{a}_1$, and $\mathbf{a}_2$ are shown in Figure 2.3.

[3] We firmly intend to avoid confusing r and $\mathbf{a}_r$ with R and $\mathbf{a}_R$. The first two refer specifically to the spherical coordinate system, whereas R and $\mathbf{a}_R$ do not refer to any coordinate system—the choice is still available to us.

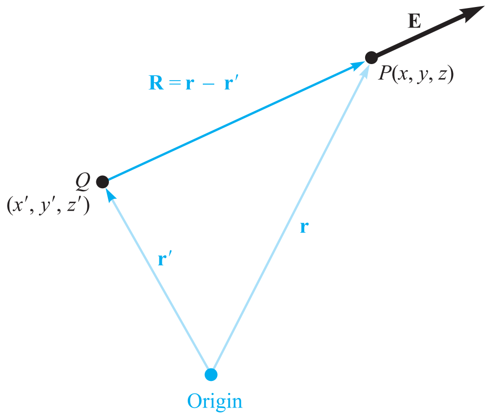

Figure 2.2 The vector $\mathbf{r}'$ locates the point charge Q, the vector $\mathbf{r}$ identifies the general point in space $P(x, y, z)$, and the vector $\mathbf{R}$ from Q to $P(x, y, z)$ is then $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

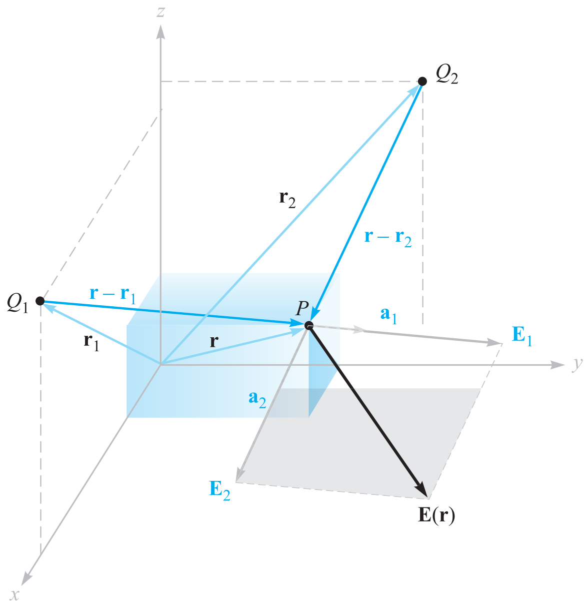

Figure 2.3 The vector addition of the total electric field intensity at P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

If we add more charges at other positions, the field due to n point charges is

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$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m \tag{11}$$

EXAMPLE 2.2

In order to illustrate the application of (11), we find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r}-\mathbf{r}_1| = 1$, $|\mathbf{r}-\mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r}-\mathbf{r}_3| = 3$, and $|\mathbf{r}-\mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96$ V · m, we may now use (11) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \text{ V/m}$$

D2.2. A charge of $-0.3\,\mu$C is located at $A(25, -30, 15)$ (in cm), and a second charge of $0.5\,\mu$C is at $B(-10, 8, 12)$ cm. Find $\mathbf{E}$ at: (*a*) the origin; (*b*) $P(15, 20, 50)$ cm.

Ans. $92.3\mathbf{a}_x - 77.6\mathbf{a}_y - 94.2\mathbf{a}_z$ kV/m; $11.9\mathbf{a}_x - 0.519\mathbf{a}_y + 12.4\mathbf{a}_z$ kV/m

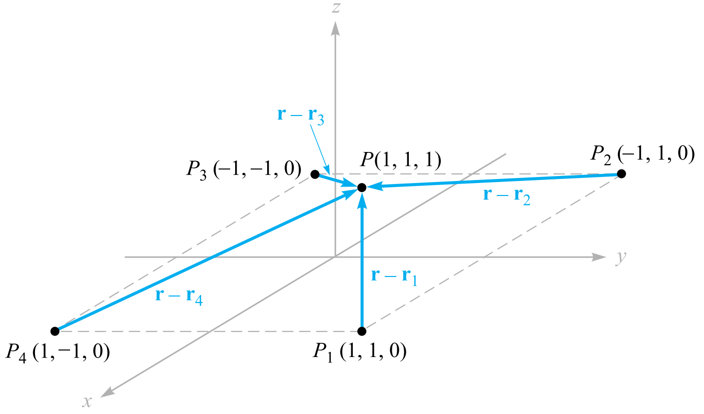

Figure 2.4 A symmetrical distribution of four identical 3-nC point charges produces a field at P, $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

D2.3. Evaluate the sums: $(a) \sum_{m=0}^{5} \dfrac{1+(-1)^m}{m^2+1}$; $(b) \sum_{m=1}^{4} \dfrac{(0.1)^m+1}{(4+m^2)^{1.5}}$

Ans. 2.52; 0.176

2.3 FIELD ARISING FROM A CONTINUOUS VOLUME CHARGE DISTRIBUTION

If we now visualize a region of space filled with a tremendous number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density*, just as we describe water as having a density of 1 g/cm^3 (gram per cubic centimeter) even though it consists of atomic- and molecular-sized particles. We can do this only if we are uninterested in the small irregularities (or ripples) in the field as we move from electron to electron or if we care little that the mass of the water actually increases in small but finite steps as each new molecule is added.

This is really no limitation at all, because the end results for electrical engineers are almost always in terms of a current in a receiving antenna, a voltage in an electronic circuit, or a charge on a capacitor, or in general in terms of some large-scale *macroscopic* phenomenon. It is very seldom that we must know a current electron by electron.[4]

We denote volume charge density by ρ_ν, having the units of coulombs per cubic meter (C/m^3).

The small amount of charge ΔQ in a small volume $\Delta\nu$ is

$$\Delta Q = \rho_\nu \Delta\nu \tag{12}$$

and we may define ρ_ν mathematically by using a limiting process on (12),

$$\boxed{\rho_\nu = \lim_{\Delta\nu \to 0} \frac{\Delta Q}{\Delta\nu}} \tag{13}$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$\boxed{Q = \int_{\text{vol}} \rho_\nu \, d\nu} \tag{14}$$

Only one integral sign is customarily indicated, but the differential $d\nu$ signifies integration throughout a volume, and hence a triple integration.

[4] A study of the noise generated by electrons in semiconductors and resistors, however, requires just such an examination of the charge through statistical analysis.

EXAMPLE 2.3

As an example of the evaluation of a volume integral, we find the total charge contained in a 2-cm length of the electron beam shown in Figure 2.5.

Solution. From the illustration, we see that the charge density is

$$\rho_v = -5 \times 10^{-6} e^{-10^5 \rho z} \text{ C/m}^2$$

The volume differential in cylindrical coordinates is given in Section 1.8; therefore,

$$Q = \int_{0.02}^{0.04} \int_{0}^{2\pi} \int_{0}^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z} \rho \, d\rho \, d\phi \, dz$$

We integrate first with respect to ϕ because it is so easy,

$$Q = \int_{0.02}^{0.04} \int_{0}^{0.01} -10^{-5} \pi e^{-10^5 \rho z} \rho \, d\rho \, dz$$

and then with respect to z, because this will simplify the last integration with respect to ρ,

$$Q = \int_{0}^{0.01} \left(\frac{-10^{-5}\pi}{-10^5 \rho} e^{-10^5 \rho z} \rho \, d\rho \right)_{z=0.02}^{z=0.04}$$

$$= \int_{0}^{0.01} -10^{-5} \pi (e^{-2000\rho} - e^{-4000\rho}) d\rho$$

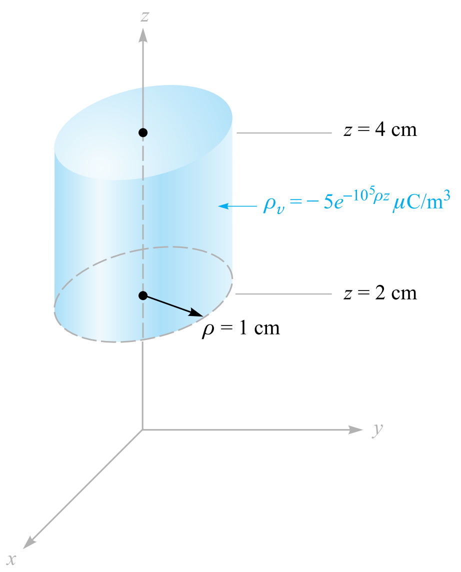

Figure 2.5 The total charge contained within the right circular cylinder may be obtained by evaluating $Q = \int_{\text{vol}} \rho_v dv$.

Finally,

$$Q = -10^{-10}\pi\left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000}\right)_0^{0.01}$$

$$Q = -10^{-10}\pi\left(\frac{1}{2000} - \frac{1}{4000}\right) = \frac{-\pi}{40} = 0.0785 \text{ pC}$$

where pC indicates picocoulombs.

The incremental contribution to the electric field intensity at $\mathbf{r}$ produced by an incremental charge ΔQ at $\mathbf{r}'$ is

$$\Delta\mathbf{E}(\mathbf{r}) = \frac{\Delta Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\rho_\nu\Delta\nu}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

If we sum the contributions of all the volume charge in a given region and let the volume element $\Delta\nu$ approach zero as the number of these elements becomes infinite, the summation becomes an integral,

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_\nu(\mathbf{r}')\,d\nu'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{15}$$

This is again a triple integral, and (except in Drill Problem 2.4) we shall do our best to avoid actually performing the integration.

The significance of the various quantities under the integral sign of (15) might stand a little review. The vector $\mathbf{r}$ from the origin locates the field point where $\mathbf{E}$ is being determined, whereas the vector $\mathbf{r}'$ extends from the origin to the source point where $\rho_\nu(\mathbf{r}')d\nu'$ is located. The scalar distance between the source point and the field point is $|\mathbf{r}-\mathbf{r}'|$, and the fraction $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ is a unit vector directed from source point to field point. The variables of integration are x', y', and z' in rectangular coordinates.

D2.4. Calculate the total charge within each of the indicated volumes: $(a)\, 0.1 \le |x|, |y|, |z| \le 0.2$: $\rho_\nu = \dfrac{1}{x^3y^3z^3}$; $(b)\, 0 \le \rho \le 0.1, 0 \le \phi \le \pi, 2 \le z \le 4$; $\rho_\nu = \rho^2 z^2 \sin 0.6\phi$; (c) universe: $\rho_\nu = e^{-2r}/r^2$.

Ans. 0; 1.018 mC; 6.28 C

2.4 FIELD OF A LINE CHARGE

Up to this point we have considered two types of charge distribution, the point charge and charge distributed throughout a volume with a density ρ_ν C/m^3. If we now consider a filamentlike distribution of volume charge density, such as a charged conductor of very small radius, we find it convenient to treat the charge as a line charge of density ρ_L C/m.

We assume a straight-line charge extending along the z axis in a cylindrical coordinate system from $-\infty$ to ∞, as shown in Figure 2.6. We desire the electric field intensity $\mathbf{E}$ at any and every point resulting from a *uniform* line charge density ρ_L.

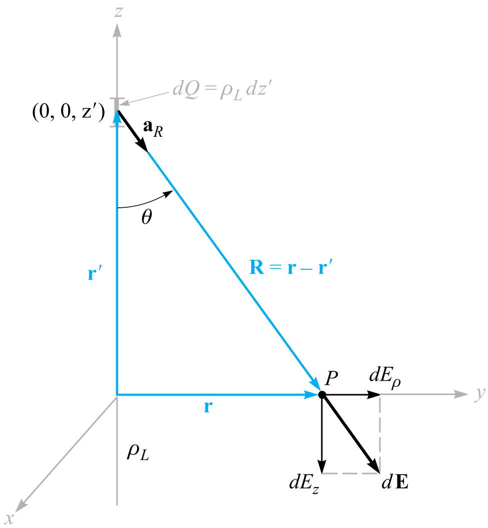

Figure 2.6 The contribution $d\mathbf{E} = dE_\rho \mathbf{a}_\rho + dE_z \mathbf{a}_z$ to the electric field intensity produced by an element of charge $dQ = \rho_L dz'$ located a distance z' from the origin. The linear charge density is uniform and extends along the entire z axis.

Symmetry should always be considered first in order to determine two specific factors: (1) with which coordinates the field does *not* vary, and (2) which components of the field are *not* present. The answers to these questions then tell us which components are present and with which coordinates they *do* vary.

Referring to Figure 2.6, we realize that as we move around the line charge, varying ϕ while keeping ρ and z constant, the line charge appears the same from every angle. In other words, azimuthal symmetry is present, and no field component may vary with ϕ.

Again, if we maintain ρ and ϕ constant while moving up and down the line charge by changing z, the line charge still recedes into infinite distance in both directions and the problem is unchanged. This is axial symmetry and leads to fields that are not functions of z.

If we maintain ϕ and z constant and vary ρ, the problem changes, and Coulomb's law leads us to expect the field to become weaker as ρ increases. Hence, by a process of elimination we are led to the fact that the field varies only with ρ.

Now, which components are present? Each incremental length of line charge acts as a point charge and produces an incremental contribution to the electric field intensity which is directed away from the bit of charge (assuming a positive line charge). No element of charge produces a ϕ component of electric intensity; E_ϕ is zero. However, each element does produce an E_ρ and an E_z component, but the contribution to E_z by elements of charge that are equal distances above and below the point at which we are determining the field will cancel.

We therefore have found that we have only an E_ρ component and it varies only with ρ. Now to find this component.

We choose a point $P(0, y, 0)$ on the y axis at which to determine the field. This is a perfectly general point in view of the lack of variation of the field with ϕ and z. Applying (10) to find the incremental field at P due to the incremental charge $dQ = \rho_L dz'$, we have

$$d\mathbf{E} = \frac{\rho_L dz'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where

$$\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$$
$$\mathbf{r}' = z'\mathbf{a}_z$$

and

$$\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

Therefore,

$$d\mathbf{E} = \frac{\rho_L dz'(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Because only the $\mathbf{E}_\rho$ component is present, we may simplify:

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

and

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Integrating by integral tables or change of variable, $z' = \rho \cot\theta$, we have

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0}\rho \left(\frac{1}{\rho^2}\frac{z'}{\sqrt{\rho^2 + z'^2}}\right)_{-\infty}^{\infty}$$

and

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

or finally,

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho} \qquad (16)$$

We note that the field falls off inversely with the distance to the charged line, as compared with the point charge, where the field decreased with the *square* of the distance. Moving ten times as far from a point charge leads to a field only 1 percent the previous strength, but moving ten times as far from a line charge only reduces the field to 10 percent of its former value. An analogy can be drawn with a source of

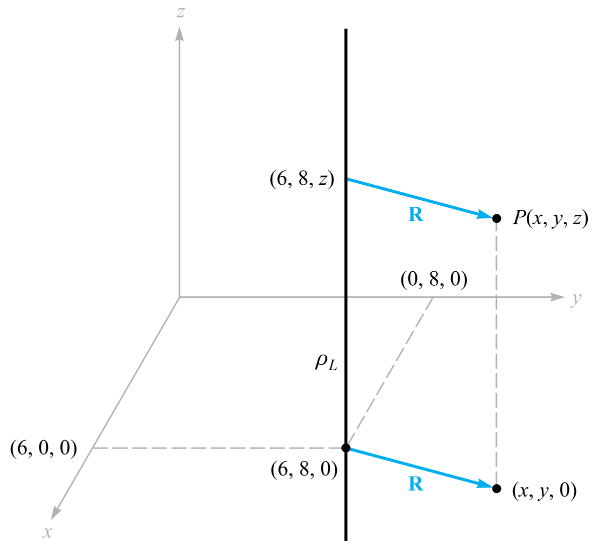

Figure 2.7 A point $P(x, y, z)$ is identified near an infinite uniform line charge located at $x = 6$, $y = 8$.

illumination, for the light intensity from a point source of light also falls off inversely as the square of the distance to the source. The field of an infinitely long fluorescent tube thus decays inversely as the first power of the radial distance to the tube, and we should expect the light intensity about a finite-length tube to obey this law near the tube. As our point recedes farther and farther from a finite-length tube, however, it eventually looks like a point source, and the field obeys the inverse-square relationship.

Before leaving this introductory look at the field of the infinite line charge, we should recognize the fact that not all line charges are located along the z axis. As an example, let us consider an infinite line charge parallel to the z axis at $x = 6$, $y = 8$, shown in Figure 2.7. We wish to find $\mathbf{E}$ at the general field point $P(x, y, z)$.

We replace ρ in (16) by the radial distance between the line charge and point, P, $R = \sqrt{(x-6)^2 + (y-8)^2}$, and let $\mathbf{a}_\rho$ be $\mathbf{a}_R$. Thus,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\sqrt{(x-6)^2 + (y-8)^2}}\mathbf{a}_R$$

where

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

Therefore,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0} \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{(x-6)^2 + (y-8)^2}$$

We again note that the field is not a function of z.

In Section 2.6, we describe how fields may be sketched, and we use the field of the line charge as one example.

D2.5. Infinite uniform line charges of 5 nC/m lie along the (positive and negative) x and y axes in free space. Find $\mathbf{E}$ at: (a) $P_A(0, 0, 4)$; (b) $P_B(0, 3, 4)$.

Ans. $45\mathbf{a}_z$ V/m; $10.8\mathbf{a}_y + 36.9\mathbf{a}_z$ V/m

2.5 FIELD OF A SHEET OF CHARGE

Another basic charge configuration is the infinite sheet of charge having a uniform density of ρ_S C/m^2. Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor. As we shall see in Chapter 5, static charge resides on conductor surfaces and not in their interiors; for this reason, ρ_S is commonly known as *surface charge density*. The charge-distribution family now is complete—point, line, surface, and volume, or Q, ρ_L, ρ_S, and ρ_v.

Let us place a sheet of charge in the yz plane and again consider symmetry (Figure 2.8). We see first that the field cannot vary with y or with z, and then we see that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we evaluate the field will cancel. Hence only E_x is present, and this component is a function of x alone. We are again faced with a choice of many methods by which to evaluate this component, and this time we use only one method and leave the others as exercises for a quiet Sunday afternoon.

Let us use the field of the infinite line charge (16) by dividing the infinite sheet into differential-width strips. One such strip is shown in Figure 2.8. The line charge

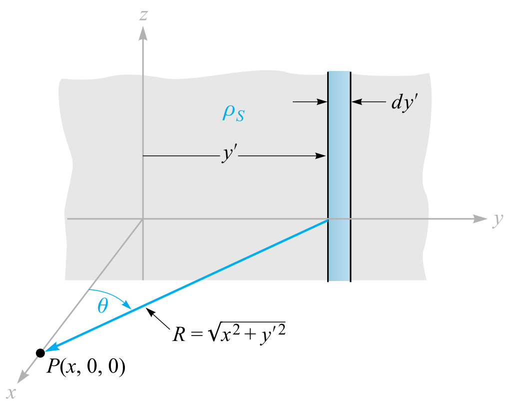

Figure 2.8 An infinite sheet of charge in the yz plane, a general point P on the x axis, and the differential-width line charge used as the element in determining the field at P by $d\mathbf{E} = \rho_S dy' \mathbf{a}_R / (2\pi\varepsilon_0 R)$.

density, or charge per unit length, is $\rho_L = \rho_S\, dy'$, and the distance from this line charge to our general point P on the x axis is $R = \sqrt{x^2 + y'^2}$. The contribution to E_x at P from this differential-width strip is then

$$dE_x = \frac{\rho_S\, dy'}{2\pi\epsilon_0\sqrt{x^2 + y'^2}} \cos\theta = \frac{\rho_S}{2\pi\epsilon_0}\,\frac{x\,dy'}{x^2 + y'^2}$$

Adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x\,dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0}\tan^{-1}\frac{y'}{x}\bigg]_{-\infty}^{\infty} = \frac{\rho_S}{2\epsilon_0}$$

If the point P were chosen on the negative x axis, then

$$E_x = -\frac{\rho_S}{2\epsilon_0}$$

for the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector $\mathbf{a}_N$, which is normal to the sheet and directed outward, or away from it. Then

$$\boxed{\mathbf{E} = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_N} \tag{17}$$

This is a startling answer, for the field is constant in magnitude and direction. It is just as strong a million miles away from the sheet as it is right off the surface. Returning to our light analogy, we see that a uniform source of light on the ceiling of a very large room leads to just as much illumination on a square foot on the floor as it does on a square foot a few inches below the ceiling. If you desire greater illumination on this subject, it will do you no good to hold the book closer to such a light source.

If a second infinite sheet of charge, having a *negative* charge density $-\rho_S$, is located in the plane $x = a$, we may find the total field by adding the contribution of each sheet. In the region $x > a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = -\frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and for $x < 0$,

$$\mathbf{E}_+ = -\frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and when $0 < x < a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_x$$

and

$$\boxed{\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho_S}{\epsilon_0}\mathbf{a}_x} \tag{18}$$

This is an important practical answer, for it is the field between the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation and provided also that we are considering a point well removed from the edges. The field outside the capacitor, while not zero, as we found for the preceding ideal case, is usually negligible.

D2.6. Three infinite uniform sheets of charge are located in free space as follows: 3 nC/m^2 at $z = -4$, 6 nC/m^2 at $z = 1$, and -8 nC/m^2 at $z = 4$. Find $\mathbf{E}$ at the point: (*a*) $P_A(2, 5, -5)$; (*b*) $P_B(4, 2, -3)$; (*c*) $P_C(-1, -5, 2)$; (*d*) $P_D(-2, 4, 5)$.

Ans. $-56.5\mathbf{a}_z$; $283\mathbf{a}_z$; $961\mathbf{a}_z$; $56.5\mathbf{a}_z$ all V/m

2.6 STREAMLINES AND SKETCHES OF FIELDS

We now have vector equations for the electric field intensity resulting from several different charge configurations, and we have had little difficulty in interpreting the magnitude and direction of the field from the equations. Unfortunately, this simplicity cannot last much longer, for we have solved most of the simple cases and our new charge distributions must lead to more complicated expressions for the fields and more difficulty in visualizing the fields through the equations. However, it is true that one picture would be worth about a thousand words, if we just knew what picture to draw.

Consider the field about the line charge,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

Figure 2.9*a* shows a cross-sectional view of the line charge and presents what might be our first effort at picturing the field—short line segments drawn here and there having lengths proportional to the magnitude of $\mathbf{E}$ and pointing in the direction of $\mathbf{E}$. The figure fails to show the symmetry with respect to ϕ, so we try again in Figure 2.9*b* with a symmetrical location of the line segments. The real trouble now appears—the longest lines must be drawn in the most crowded region, and this also plagues us if we use line segments of equal length but of a thickness that is proportional to $\mathbf{E}$ (Figure 2.9*c*). Other schemes include drawing shorter lines to represent stronger fields (inherently misleading) and using intensity of color or different colors to represent stronger fields.

For the present, let us be content to show only the *direction* of $\mathbf{E}$ by drawing continuous lines, which are everywhere tangent to $\mathbf{E}$, from the charge. Figure 2.9*d* shows this compromise. A symmetrical distribution of lines (one every 45°) indicates azimuthal symmetry, and arrowheads should be used to show direction.

These lines are usually called *streamlines*, although other terms such as flux lines and direction lines are also used. A small positive test charge placed at any point in this field and free to move would accelerate in the direction of the streamline passing through that point. If the field represented the velocity of a liquid or a gas (which, incidentally, would have to have a source at $\rho = 0$), small suspended particles in the liquid or gas would trace out the streamlines.

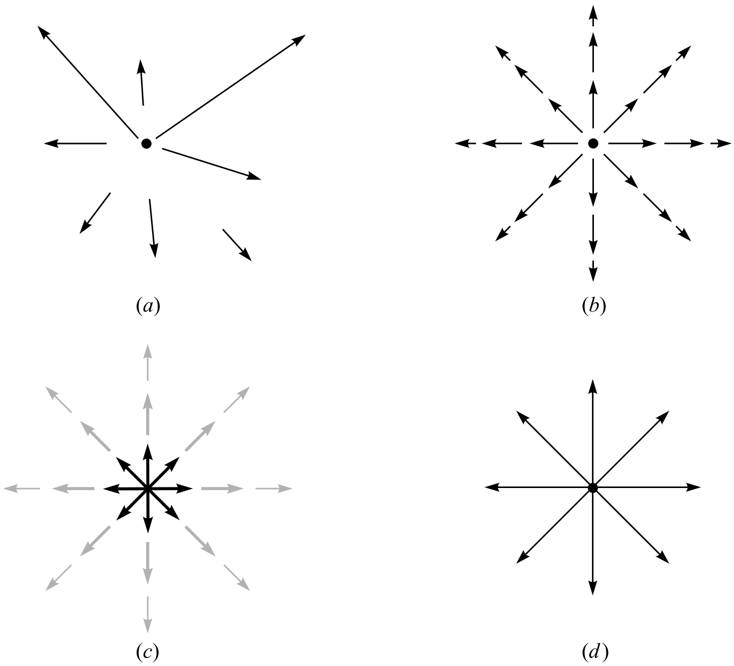

Figure 2.9 (a) One very poor sketch, (b) and (c) two fair sketches, and (d) the usual form of a streamline sketch. In the last form, the arrows show the direction of the field at every point along the line, and the spacing of the lines is inversely proportional to the strength of the field.

We will find out later that a bonus accompanies this streamline sketch, for the magnitude of the field can be shown to be inversely proportional to the spacing of the streamlines for some important special cases. The closer they are together, the stronger is the field. At that time we will also find an easier, more accurate method of making that type of streamline sketch.

If we attempted to sketch the field of the point charge, the variation of the field into and away from the page would cause essentially insurmountable difficulties; for this reason sketching is usually limited to two-dimensional fields.

In the case of the two-dimensional field, let us arbitrarily set $E_z = 0$. The streamlines are thus confined to planes for which z is constant, and the sketch is the same for any such plane. Several streamlines are shown in Figure 2.10, and the E_x and E_y components are indicated at a general point. It is apparent from the geometry that

$$\boxed{\frac{E_y}{E_x} = \frac{dy}{dx}} \tag{19}$$

A knowledge of the functional form of E_x and E_y (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

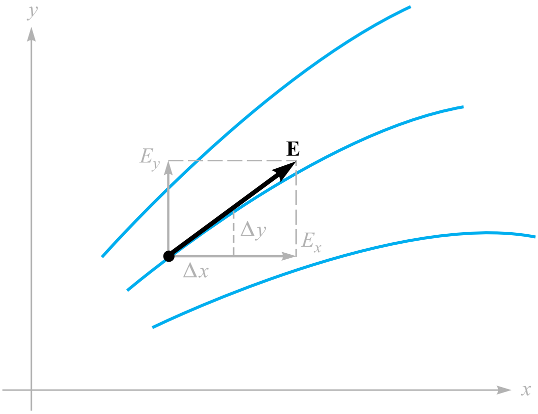

Figure 2.10 The equation of a streamline is obtained by solving the differential equation $E_y/E_x = dy/dx$.

As an illustration of this method, consider the field of the uniform line charge with $\rho_L = 2\pi\epsilon_0$,

$$\mathbf{E} = \frac{1}{\rho}\mathbf{a}_\rho$$

In rectangular coordinates,

$$\mathbf{E} = \frac{x}{x^2+y^2}\mathbf{a}_x + \frac{y}{x^2+y^2}\mathbf{a}_y$$

Thus we form the differential equation

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x} \qquad \text{or} \qquad \frac{dy}{y} = \frac{dx}{x}$$

Therefore,

$$\ln y = \ln x + C_1 \qquad \text{or} \qquad \ln y = \ln x + \ln C$$

from which the equations of the streamlines are obtained,

$$y = Cx$$

If we want to find the equation of one particular streamline, say one passing through $P(-2, 7, 10)$, we merely substitute the coordinates of that point into our equation and evaluate C. Here, $7 = C(-2)$, and $C = -3.5$, so $y = -3.5x$.

Each streamline is associated with a specific value of C, and the radial lines shown in Figure 2.9*d* are obtained when $C = 0, 1, -1$, and $1/C = 0$.

The equations of streamlines may also be obtained directly in cylindrical or spherical coordinates. A spherical coordinate example will be examined in Section 4.7.

D2.7. Find the equation of that streamline that passes through the point $P(1, 4, -2)$ in the field $\mathbf{E} =$ (*a*) $\dfrac{-8x}{y}\mathbf{a}_x + \dfrac{4x^2}{y^2}\mathbf{a}_y$; (*b*) $2e^{5x}[y(5x+1)\mathbf{a}_x + x\mathbf{a}_y]$.

Ans. $x^2 + 2y^2 = 33$; $y^2 = 15.7 + 0.4x - 0.08\ln(5x+1)$

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2. Della Torre, E., and Longo, C. L. *The Electromagnetic Field*. Boston: Allyn and Bacon, 1969. The authors introduce all of electromagnetic theory with a careful and rigorous development based on a single experimental law—that of Coulomb. It begins in Chapter 1.
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CHAPTER 2 PROBLEMS

2.1 Three point charges are positioned in the x-y plane as follows: 5 nC at $y = 5$ cm, -10 nC at $y = -5$ cm, and 15 nC at $x = -5$ cm. Find the required x-y coordinates of a 20-nC fourth charge that will produce a zero electric field at the origin.

2.2 Point charges of 1 nC and -2 nC are located at (0, 0, 0) and (1, 1, 1), respectively, in free space. Determine the vector force acting on each charge.

2.3 Point charges of 50 nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A.

2.4 Eight identical point charges of Q C each are located at the corners of a cube of side length a, with one charge at the origin, and with the three nearest charges at $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$. Find an expression for the total vector force on the charge at $P(a, a, a)$, assuming free space.

2.5 Let a point charge $Q_1 = 25$ nC be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3, 4, -2)$. (*a*) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1, 2, 3)$. (*b*) At what point on the y axis is $E_x = 0$?

2.6 Two point charges of equal magnitude q are positioned at $z = \pm d/2$. (*a*) Find the electric field everywhere on the z axis; (*b*) find the electric field everywhere on the x axis; (*c*) repeat parts (*a*) and (*b*) if the charge at $z = -d/2$ is $-q$ instead of $+q$.

2.7 A 2-μC point charge is located at $A(4, 3, 5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8, 12, 2)$.

2.8 A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x axis and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q/C, obtain the expression by which Q may be found as a function of x. Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and state the separation of the spheres then. What happens if a larger charge is applied?

2.9 A 100-nC point charge is located at $A(-1, 1, 3)$ in free space. (*a*) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m. (*b*) Find y_1 if $P(-2, y_1, 3)$ lies on that locus.

2.10 A charge of -1 nC is located at the origin in free space. What charge must be located at (2, 0, 0) to cause E_x to be zero at (3, 1, 1)?

2.11 A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$. (*a*) Find Q_0. Find $\mathbf{E}$ at $M(1, 6, 5)$ in (*b*) rectangular coordinates; (*c*) cylindrical coordinates; (*d*) spherical coordinates.

2.12 Electrons are in random motion in a fixed region in space. During any 1 μs interval, the probability of finding an electron in a subregion of volume 10^{-15} m^2 is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

2.13 A uniform volume charge density of 0.2 μC/m^3 is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere, find (*a*) the total charge present throughout the shell, and (*b*) r_1 if half the total charge is located in the region 3 cm $< r < r_1$.

2.14 The electron beam in a certain cathode ray tube possesses cylindrical symmetry, and the charge density is represented by $\rho_v = -0.1/(\rho^2 + 10^{-8})$ pC/m^3 for $0 < \rho < 3 \times 10^{-4}$ m, and $\rho_v = 0$ for $\rho > 3 \times 10^{-4}$ m. (*a*) Find the total charge per meter along the length of the beam; (*b*) if the electron velocity is 5×10^7 m/s, and with one ampere defined as 1C/s, find the beam current.

2.15 A spherical volume having a 2-μm radius contains a uniform volume charge density of 10^{15} C/m^3. (*a*) What total charge is enclosed in the spherical volume? (*b*) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3 mm on a side and that there is no charge between the spheres. What is the average volume charge density throughout this large region?

2.16 Within a region of free space, charge density is given as $\rho_v = \frac{\rho_0 r \cos\theta}{a}$ C/m^3, where ρ_0 and a are constants. Find the total charge lying within (*a*) the sphere, $r \leq a$; (*b*) the cone, $r \leq a$, $0 \leq \theta \leq 0.1\pi$; (*c*) the region, $r \leq a$, $0 \leq \theta \leq 0.1\pi$, $0 \leq \phi \leq 0.2\pi$.

2.17 A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$: (*a*) find **E** at $P(1, 2, 3)$. (*b*) find **E** at that point in the $z = 0$ plane where the direction of **E** is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$.

2.18 (a) Find **E** in the plane $z = 0$ that is produced by a uniform line charge, ρ_L, extending along the z axis over the range $-L < z < L$ in a cylindrical coordinate system. (b) If the finite line charge is approximated by an infinite line charge ($L \to \infty$), by what percentage is E_ρ in error if $\rho = 0.5L$? (*c*) Repeat (*b*) with $\rho = 0.1L$.

2.19 A uniform line charge of 2 μC/m is located on the z axis. Find **E** in rectangular coordinates at $P(1, 2, 3)$ if the charge exists from (*a*) $-\infty < z < \infty$; (*b*) $-4 \le z \le 4$.

2.20 A line charge of uniform charge density ρ_0 C/m and of length ℓ is oriented along the z axis at $-\ell/2 < z < \ell/2$. (*a*) Find the electric field strength, **E**, in magnitude and direction at any position along the x axis. (*b*) With the given line charge in position, find the force acting on an identical line charge that is oriented along the x axis at $\ell/2 < x < 3\ell/2$.

2.21 Two identical uniform line charges, with $\rho_l = 75$ nC/m, are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other?

2.22 Two identical uniform sheet charges with $\rho_s = 100$ nC/m^2 are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

2.23 Given the surface charge density, $\rho_s = 2$ μC/m^2, existing in the region $\rho < 0.2$ m, $z = 0$, find **E** at (*a*) $P_A(\rho = 0, z = 0.5)$; (*b*) $P_B(\rho = 0, z = -0.5)$. Show that (*c*) the field along the z axis reduces to that of an infinite sheet charge at small values of z; (*d*) the z axis field reduces to that of a point charge at large values of z.

2.24 (*a*) Find the electric field on the z axis produced by an annular ring of uniform surface charge density ρ_s in free space. The ring occupies the region $z = 0$, $a \le \rho \le b$, $0 \le \phi \le 2\pi$ in cylindrical coordinates. (*b*) From your part (*a*) result, obtain the field of an infinite uniform sheet charge by taking appropriate limits.

2.25 Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC, at $P(2, 0, 6)$; uniform line charge density, 3 nC/m, at $x = -2$, $y = 3$; uniform surface charge density, 0.2 nC/m^2 at $x = 2$.

2.26 A radially dependent surface charge is distributed on an infinite flat sheet in the x-y plane and is characterized in cylindrical coordinates by surface density $\rho_s = \rho_0/\rho$, where ρ_0 is a constant. Determine the electric field strength, **E**, everywhere on the z axis.

2.27 Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find (*a*) the equation of the streamline that passes through the point $P(2, 3, -4)$; (*b*) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$.

2.28 An electric dipole (discussed in detail in Section 4.7) consists of two point charges of equal and opposite magnitude $\pm Q$ spaced by distance d. With the charges along the z axis at positions $z = \pm d/2$ (with the positive charge at the positive z location), the electric field in spherical coordinates is given by $\mathbf{E}(r, \theta) = [Qd/(4\pi\epsilon_0 r^3)][2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude q (*a*) at $(0, 0, z)$; (*b*) at $(0, y, 0)$.

2.29 If $\mathbf{E} = 20e^{-5y}(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, find (*a*) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$; (*b*) a unit vector in the direction of $\mathbf{E}$ at P; (*c*) the equation of the direction line passing through P.

2.30 For fields that do not vary with z in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_p/E_\phi = d\rho/(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30^\circ, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\mathbf{a}_\rho - \rho\sin 2\phi\mathbf{a}_\phi$.